Confidence sets in nonparametric calibration of exponential Lévy models

Jakob Söhl*
Humboldt-Universität zu Berlin

March 1, 2012

Abstract

Confidence intervals and joint confidence sets are constructed for the nonparametric calibration of exponential Lévy models based on prices of European options. This is done by showing joint asymptotic normality for the estimation of the volatility, the drift, the intensity and the Lévy density at finitely many points in the spectral calibration method. Furthermore, the asymptotic normality result leads to a test on the value of the volatility in exponential Lévy models.

Keywords: European option · Jump diffusion · Confidence sets · Asymptotic normality · Nonlinear inverse problem

MSC (2000): 60G51 · 62G10 · 62G15 · 91B28

JEL Classification: G13 · C14

1 Introduction

The unknown future development of financial markets faced by its participants, whether as investors, as traders or as companies, can be understood to consist of model risk and "Knightian uncertainty" [10, 17]. The first describes the risk for a given calibrated model and can be evaluated by probabilistic methods, whereas the second incorporates the lack of knowledge on the underlying probability measure and is typically treated by worst case scenarios, for example, by stress testing, which amounts to taking the supremum or infimum over a range of probability measures.

*I thank Markus Reiß for fruitful discussions at all stages of this work and Mathias Trabs for helpful comments on the manuscript. This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk". E-mail address: soehl@math.hu-berlin.de

By the choice of the model, there is a trade-off in the uncertainty between the calibration error and the misspecification of the model, where larger models reduce the latter. Since the calibration error is statistically traceable and the misspecification is hard to track, we choose a rich model such as a nonparametric one and then asses the calibration error by constructing confidence sets.

More precisely, we consider the nonparametric calibration when the risk-neutral price of a stock (S_t) follows an exponential Lévy model

$$S_t = Se^{rt+X_t} \quad \text{with a Lévy process } X_t \text{ for } t \geq 0. \tag{1.1}$$

A thorough discussion of this model is given in the monograph by Cont and Tankov [7]. They introduced in [8, 9] a nonparametric calibration method for this model based on prices of European call and put options, in which a least squares approach is penalized by relative entropy. Belomestny and Reiß [2] used a different approach to the same estimation problem, where they regularized by a spectral cut-off and constructed estimators that achieve the minimax rates of convergence. We show asymptotic normality and construct confidence sets and intervals for their estimation procedure. Methods similar to theirs were also applied by Belomestny [1] to estimate the fractional order of regular Lévy processes of exponential type and by Trabs [22] to estimate self-decomposable Lévy processes.

Confidence sets measure how reliable the estimation is. This is particularly important if the calibrated model is to be used for pricing and hedging. For a recent review on pricing and hedging in exponential Lévy models see [21] and the references therein. For the influence of model uncertainty on the pricing see [6].

Nonparametric confidence intervals and sets for Lévy triplets have not been studied with the notable exception of the work by Figueroa-López [12]. The work is more general in the sense that beyond pointwise confidence intervals also confidence bands are constructed. On the other hand, the method is based on high-frequency observations so that the statistical problem of estimating the Lévy density reduces to the problem of density estimation. In contrast to this direct observation scheme our method is based on option prices and thus the calibration is a nonlinear inverse problem, which is mildly ill-posed for volatility zero and severely ill-posed for positive volatility.

Confidence intervals and sets in nonparametric problems are a subtle issue. Whether adaptive confidence intervals for the estimators of the volatility, the drift and the intensity exist is an open question. We show asymptotic normality for the parametric estimators of the volatility, the drift and the intensity. We also proof asymptotic normality for the pointwise estimators of the Lévy density. The joint asymptotic distribution of these estimators is derived in both the mildly and the severely ill-posed case. This is used for the construction of confidence intervals and joint confidence sets as well

as for a test on the value of the volatility. The asymptotic normality results are based on undersmoothing and on a linearization of the stochastic errors.

The paper is organized as follows. The model and the estimation method are given in Section 2. The main results are formulated in Section 3. They are applied to confidence intervals and to a hypotheses test on the value of the volatility in Section 4. We conclude in Section 5. Proofs are deferred to Section 6 and Section 7. Uniform convergence is treated in Section 8.

2 The model and the estimators

We denote by $\mathcal{C}(K,T)$ and $\mathcal{P}(K,T)$ prices of European call and put options on the underlying (S_t) with strike price K and maturity T. We suppose that the risk-neutral price of the stock (S_t) follows the exponential Lévy model (1.1) with respect to an equivalent martingale measure $\mathbb{Q}$, where $S>0$ is the present value of the stock and $r\geq 0$ is the riskless interest rate. We fix some T and assume that the observations are given by option prices for different maturities (K_j) corrupted by noise:

$$Y_j=\mathcal{C}(K_j,T)+\sigma_j\epsilon_j,\quad j=1,\dots,n.$$

The noise levels (σ_j) are assumed to be positive and known. The minimax result in [2] is shown for general errors (ϵ_j) which are independent centered random variables with $\mathbb{E}[\epsilon_j^2]=1$ and $\sup_j\mathbb{E}[\epsilon_j^4]<\infty$. We transform the observations to a regression problem on the function

$$\mathcal{O}(x):=\begin{cases} S^{-1}\mathcal{C}(x,T), & x\geq 0,\\ S^{-1}\mathcal{P}(x,T), & x<0,\end{cases}$$

where $x:=\log(K/S)-rT$ denotes the log-forward moneyness. The regression model may then be written as

$$O_j=\mathcal{O}(x_j)+\delta_j\epsilon_j. \tag{2.1}$$

We call the volatility of a Lévy process σ, the drift γ and the intensity λ. As in [2] we consider only Lévy processes (X_t) with a jump component of finite intensity and absolutely continuous jump distribution. For a jump density ν we denote by $\mu(x):=e^x\nu(x)$ the corresponding exponentially weighted jump density. The aim is to estimate the Lévy triplet $\mathcal{T}=(\sigma^2,\gamma,\mu)$. In the remainder of this section we present the spectral calibration method of Belomestny and Reiß [2]. The method is based on a pricing formula by Carr and Madan [5], which relates the Fourier transform $\mathcal{F}\mathcal{O}(u):=\int_{-\infty}^{\infty}\mathcal{O}(x)e^{iux}\mathrm{d}x$ to the characteristic function $\varphi_T(u):=\mathbb{E}[e^{iuX_T}]$. That is why we define

$$\begin{aligned}\psi(u)&:=\frac{1}{T}\log\left(1+iu(1+iu)\mathcal{F}\mathcal{O}(u)\right)=\frac{1}{T}\log(\varphi_T(u-i))\\ &=-\frac{\sigma^2u^2}{2}+i(\sigma^2+\gamma)u+(\sigma^2/2+\gamma-\lambda)+\mathcal{F}\mu(u),\end{aligned} \tag{2.2}$$

where the first equality is given by the above mentioned pricing formula and the second by the Lévy-Khintchine representation. This equation links the observation of $\mathcal{O}$ to the Lévy triplet that we want to estimate. Let $\mathcal{O}_\epsilon$ be an approximation on the true function $\mathcal{O}$. For example, $\mathcal{O}_\epsilon$ can be obtained by linear interpolation of the data (2.1). We further define the empirical counterpart of ψ by

$$\psi_\epsilon(u) := \frac{1}{T} \log_{\geq \kappa(u)} \left(1 + iu(1+iu)\mathcal{F}\mathcal{O}_\epsilon(u)\right),$$

where the trimmed logarithm $\log_{\geq\kappa} : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ is given by

$$\log_{\geq\kappa}(z) := \begin{cases} \log(z), & |z| \geq \kappa \\ \log(\kappa\, z/|z|), & |z| < \kappa \end{cases}.$$

The logarithms are taken in such a way that ψ and ψ_ϵ are continuous with $\psi(0) = \psi_\epsilon(0) = 0$ and $\kappa(u) \in (0,1)$ is specified in [2]. Considering (2.2) as a quadratic polynomial in u disturbed by $\mathcal{F}\mu$ motivates the following definition of the estimators for a cut-off value $U > 0$:

$$\hat{\sigma}^2 := \int_{-U}^{U} \mathrm{Re}(\psi_\epsilon(u)) w_\sigma^U(u) \mathrm{d}u, \tag{2.3}$$

$$\hat{\gamma} := -\hat{\sigma}^2 + \int_{-U}^{U} \mathrm{Im}(\psi_\epsilon(u)) w_\gamma^U(u) \mathrm{d}u, \tag{2.4}$$

$$\hat{\lambda} := \frac{\hat{\sigma}^2}{2} + \hat{\gamma} - \int_{-U}^{U} \mathrm{Re}(\psi_\epsilon(u)) w_\lambda^U(u) \mathrm{d}u, \tag{2.5}$$

where the weight functions w_σ^U, w_γ^U and w_λ^U satisfy

$$\begin{aligned} &\int_{-U}^{U} \frac{-u^2}{2} w_\sigma^U(u) \mathrm{d}u = 1, \quad \int_{-U}^{U} u w_\gamma^U(u) \mathrm{d}u = 1, \quad \int_{-U}^{U} w_\lambda^U(u) \mathrm{d}u = 1, \\ &\int_{-U}^{U} w_\sigma^U(u) \mathrm{d}u = 0, \quad \int_{-U}^{U} u^2 w_\lambda^U(u) \mathrm{d}u = 0. \end{aligned} \tag{2.6}$$

The estimator for μ is defined by a smoothed inverse Fourier transform of the remainder

$$\hat{\mu}(x) := \mathcal{F}^{-1}\left[\left(\psi_\epsilon(u) + \frac{\hat{\sigma}^2}{2}(u-i)^2 - i\hat{\gamma}(u-i) + \hat{\lambda}\right) w_\mu^U(u)\right](x). \tag{2.7}$$

The weight functions for all $U > 0$ can be obtained from w_σ^1, w_γ^1, w_λ^1 and w_μ^1 by rescaling:

$$\begin{aligned} w_\sigma^U(u) &= U^{-3} w_\sigma^1(u/U), \quad & w_\gamma^U(u) &= U^{-2} w_\gamma^1(u/U), \\ w_\lambda^U(u) &= U^{-1} w_\lambda^1(u/U), \quad & w_\mu^U(u) &= w_\mu^1(u/U). \end{aligned}$$

Since $\psi_\epsilon(-u) = \overline{\psi_\epsilon(u)}$ only the symmetric part of w^1_σ, w^1_λ and the antisymmetric part of w^1_γ matter. The antisymmetric part of w^1_μ contributes a purely imaginary part to $\hat{\mu}(x)$. Without loss of generality we will always assume w^1_σ, w^1_λ, w^1_μ to be symmetric and w^1_γ to be antisymmetric. We further assume that the support of w^1_σ, w^1_γ, w^1_λ and w^1_μ is contained in $[-1, 1]$. We define the estimation error $\Delta\hat{\sigma}^2 := \hat{\sigma}^2 - \sigma^2$ and likewise for the other estimators. We will also use the notation $\Delta\psi_\epsilon := \psi_\epsilon - \psi$. The estimation error $\Delta\hat{\sigma}^2$ can be decomposed as

$$\begin{aligned} \Delta\hat{\sigma}^2 :&= 2U^{-2} \int_0^1 \mathrm{Re}(\mathcal{F}\mu(Uu)) w^1_\sigma(u) \mathrm{d}u \\ &\quad + 2U^{-2} \int_0^1 \mathrm{Re}(\Delta\psi_\epsilon(Uu)) w^1_\sigma(u) \mathrm{d}u. \end{aligned} \tag{2.8}$$

The first term is the approximation error and decreases in U due to the decay of $\mathcal{F}\mu$. The second is the stochastic error and increases in U by the growth of $\Delta\psi_\epsilon$. The cut-off value U is the crucial tuning parameter in this method and allows a trade-off between the error terms. The other estimation errors allow similar decompositions as $\Delta\hat{\sigma}^2$ in (2.8).

We shall analyze the asymptotic properties of the stochastic errors in depth. To bound the approximation errors some smoothness assumption is necessary. We assume that the Lévy triplet belongs to a smoothness class $\mathcal{G}_s(R, \sigma_{\max})$ with $s \in \mathbb{N}$ and $R, \sigma_{\max} > 0$ specified in [2, Definition 4.1]. The assumption $\mathcal{T} \in \mathcal{G}_s(R, \sigma_{\max})$ includes a smoothness assumption of order s on μ leading to a decay of $\mathcal{F}\mu$. To profit from this decay when bounding the approximation error, we assume the weight functions to be of order s, this means

$$\mathcal{F}(w^1_\sigma(u)/u^s), \mathcal{F}(w^1_\gamma(u)/u^s), \mathcal{F}(w^1_\gamma(u)/u^s), \mathcal{F}\left((1 - w^1_\mu(u))/u^s\right) \in L^1(\mathbb{R}). \tag{2.9}$$

3 Asymptotic normality

3.1 The main results

The aim of this section is to establish asymptotic normality results for the estimators. We would like to state that the appropriately scaled errors of the estimators converge to normal random variables. The starting point of the error analysis is the decomposition (2.8) into the approximation error and the stochastic error. The approximation error is deterministic and only the stochastic error can be expected to converge with appropriate scaling to a normal random variable. It is common practice to resolve this problem by undersmoothing, which means that the tuning parameter is

chosen such that the approximation error becomes asymptotically negligible. Thus the cut-off value has to grow fast enough. This is ensured by the condition $\epsilon U(\epsilon)^{(2s+5)/2} \to \infty$ in the case of volatility zero and by $\epsilon U(\epsilon)^{s+1} \exp(T\sigma^2 U(\epsilon)^2/2) \to \infty$ in the case of positive volatility.

Since the approximation errors are negligible by these conditions we focus on the stochastic errors. To simplify the asymptotic analysis we do not work with the regression model (2.1) but with the Gaussian white noise model. This is an idealized observation scheme, where the terms are easier to analyze. At the same time asymptotic results may be transferred to the regression model. The Gaussian white noise model is given by

$$\mathrm{d}\mathcal{O}_\epsilon(x) = \mathcal{O}(x)\mathrm{d}x + \epsilon\, \delta(x)\mathrm{d}W(x), \tag{3.1}$$

where W is a two-sided Brownian motion, $\delta \in L^2(\mathbb{R})$ and $\epsilon > 0$. Here the empirical counterpart $\mathcal{O}_\epsilon$ of $\mathcal{O}$ is given directly by the model and no further approximation is necessary. Transferring asymptotic results from the Gaussian white noise model to the regression model is formally justified by the concept of asymptotic equivalence. Due to the asymptotic equivalence between regression with regular errors and regression with Gaussian errors in [14] we will consider for our asymptotic analysis only the case of Gaussian errors. The asymptotic equivalence of regression (2.1) with Gaussian errors and the Gaussian white noise model (3.1) is given in Brown and Low [4], where $\delta_j = \delta(x_j)$ and ϵ corresponds to $1/\sqrt{n}$ up to a logarithmic factor. Further assumptions for the asymptotic equivalence to hold are given in Section 6.

The stochastic errors involve the term $\Delta\psi_\epsilon(Uu)$, which is a difference between two logarithms. For $z, z' \in \mathbb{C}\backslash\{0\}$ and $\kappa > 0$ it holds $\log_{\geq\kappa}(z) - \log(z') = \log_{\geq\kappa/|z'|}(z/z')$. That yields

$$\begin{aligned}
&\Delta\psi_\epsilon(Uu) \\
&= \frac{1}{T}\log_{\geq\kappa^U(u)}\Big(1 + \frac{\epsilon\, iUu(1+iUu)}{1+iUu(1+iUu)\mathcal{F}\mathcal{O}(Uu)}\int_{-\infty}^{\infty} e^{iUux}\delta(x)\mathrm{d}W(x)\Big),
\end{aligned}$$

where $\kappa^U(u) := \kappa(Uu)/|1+iUu(1+iUu)\mathcal{F}\mathcal{O}(Uu)| \leq 1/2$, see [2, (6.3)]. We define a linearization $\mathcal{L}_{\epsilon,U}$ of the logarithm and the remainder term $\mathcal{R}_{\epsilon,U}$ by

$$\mathcal{L}_{\epsilon,U}(u) := \frac{\epsilon\, iUu(1+iUu)}{T(1+iUu(1+iUu)\mathcal{F}\mathcal{O}(Uu))}\int_{-\infty}^{\infty} e^{iUux}\delta(x)\mathrm{d}W(x), \tag{3.2}$$

$$\mathcal{R}_{\epsilon,U}(u) := \Delta\psi_\epsilon(Uu) - \mathcal{L}_{\epsilon,U}(u). \tag{3.3}$$

To ensure continuity of the Gaussian process $X(u) = \int_{-\infty}^{\infty} e^{iux}\delta(x)\mathrm{d}W(x)$ we assume that there is a $p > 1$ such that $\int_{-\infty}^{\infty}(1+|x|)^p\delta(x)^2\mathrm{d}x < \infty$. In [19] it is shown that on this assumption X satisfies the Kolmogorov-Chentsov criterion [16, p. 57] and thus has a continuous version. In the sequel we are always working with this version.

The remainder term $\mathcal{R}_{\epsilon,U}$ in (3.3) is small when the argument of the logarithm is close to one, that is when $\mathcal{L}_{\epsilon,U}$ is small. Since we are integrating over the unit interval in (2.8) we want $\mathcal{L}_{\epsilon,U}$ to be uniformly small. We shall use the notation $A(x) \lesssim B(x)$ as $x \to \infty$ synonymously with the Landau notation $A(x) = O(B(x))$ as $x \to \infty$, meaning that there exist $M > 0$ and $x_0 \in \mathbb{R}$ such that $A(x) \le MB(x)$ for all $x \ge x_0$.

Proposition 1. *For all $q \ge 1$ holds*

$$\mathbb{E}\left[\sup_{u\in[-1,1]} |\mathcal{L}_{\epsilon,U}(u)|^q\right]^{1/q} \lesssim \epsilon U^2 \sqrt{\log(U)} \exp(T\sigma^2 U^2/2)$$

as $U \to \infty$.

This proposition is proved in Section 6 by metric entropy arguments. In the following theorems we control the supremum of $\mathcal{L}_{\epsilon,U}$ and thus the remainder term $\mathcal{R}_{\epsilon,U}$ by the condition $\epsilon U(\epsilon)^2\sqrt{\log(U(\epsilon))}\exp(T\sigma^2 U(\epsilon)^2/2) \to 0$. Then the asymptotic distribution of the stochastic errors $\int_0^1 \Delta\psi_\epsilon(Uu)w(u)\mathrm{d}u$ is governed by the linearized stochastic errors $\int_0^1 \mathcal{L}_{\epsilon,U}(u)w(u)\mathrm{d}u$ and the remainder term $\int_0^1 \mathcal{R}_{\epsilon,U}(u)w(u)\mathrm{d}u$ is asymptotically negligible. In the case $\sigma = 0$ the stronger condition $\epsilon U(\epsilon)^{5/2} \to 0$ is assumed, which is needed for the stochastic errors to converge to zero.

In the results on asymptotic normality we will also include the estimator $\hat{\mu}(0)$ of the jump density at zero. This only makes sense by our smoothness assumption on μ since there is no way of detecting jumps of height zero. Unlike for points $x \ne 0$ it will turn out that not the weight function w_μ^1 determines the asymptotic distribution but the effective weight function

$$w_0(u) := w_\mu^1(u) + w_\sigma^1(u)\int_{-1}^1 v^2 w_\mu^1(v)\mathrm{d}v/2 - w_\lambda^1(u)\int_{-1}^1 w_\mu^1(v)\mathrm{d}v.$$

The first theorem states the joint asymptotic normality result for the mildly ill-posed case of volatility zero.

Theorem 1. *Let $\sigma = 0$. Let δ be continuous at $T\gamma, x_1 + T\gamma, \ldots, x_n + T\gamma$ and let $\mathcal{F}\delta^2 \in L^1(\mathbb{R})$. For $j = 1, \ldots, n$ let $x_j \in \mathbb{R}\backslash\{0\}$ be distinct and let $V_0, W_0, W_{x_1}\ldots, W_{x_n}$ be independent Brownian motions. If $\epsilon U(\epsilon)^{5/2} \to 0$ and $\epsilon U(\epsilon)^{(2s+5)/2} \to \infty$ as $\epsilon \to 0$, then it holds*

$$\frac{1}{\epsilon}\begin{pmatrix} U(\epsilon)^{+1/2} & \Delta\hat{\sigma}^2 \\ U(\epsilon)^{-1/2} & \Delta\hat{\gamma} \\ U(\epsilon)^{-3/2} & \Delta\hat{\lambda} \\ U(\epsilon)^{-5/2} & \Delta\hat{\mu}(0) \\ U(\epsilon)^{-5/2} & \Delta\hat{\mu}(x_1) \\ & \vdots \\ U(\epsilon)^{-5/2} & \Delta\hat{\mu}(x_n) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} d(0) & \int_0^1 u^2 w_\sigma^1(u)\mathrm{d}W_0(u) \\ d(0) & \int_0^1 u^2 w_\gamma^1(u)\mathrm{d}V_0(u) \\ d(0) & \int_0^1 u^2 w_\lambda^1(u)\mathrm{d}W_0(u) \\ d(0) & \int_0^1 u^2 w_0(u)\mathrm{d}W_0(u)/(2\pi) \\ d(x_1) & \int_0^1 u^2 w_\mu^1(u)\mathrm{d}W_{x_1}(u)/(2\pi) \\ & \vdots \\ d(x_n) & \int_0^1 u^2 w_\mu^1(u)\mathrm{d}W_{x_n}(u)/(2\pi) \end{pmatrix},$$

as $\epsilon \to 0$*, where* $d(x) := 2\sqrt{\pi}\delta(x + T\gamma)\exp(T(\lambda - \gamma))/T$*.*

Next we consider the case $\sigma > 0$. Let δ be in $L^\infty(\mathbb{R})$ and $\|\delta\|_{L^2(\mathbb{R})} > 0$. We set

$$d := \sqrt{2}\|\delta\|_{L^2(\mathbb{R})}\exp(T(\lambda - \gamma - \sigma^2/2))T^{-2}\sigma^{-2}. \tag{3.4}$$

and define real-valued random variables $W_{\epsilon,U}$ and $V_{\epsilon,U}$ by $W_{\epsilon,U} + iV_{\epsilon,U} := 2d^{-1}\int_0^1 \mathcal{L}_{\epsilon,U}(u)\mathrm{d}u$. By Lemma 3 below

$$\frac{1}{\epsilon\exp(T\sigma^2U^2/2)}\begin{pmatrix} W_{\epsilon,U} \\ V_{\epsilon,U}\end{pmatrix} \xrightarrow{d} \begin{pmatrix} W \\ V\end{pmatrix} \tag{3.5}$$

as $U \to \infty$, where W and V are independent standard normal random variables.

The following theorem treats the stochastic errors in the case of positive volatility. Since the theorem contains no statement on the approximation errors, the condition (2.9) on the order of the weight functions may be omitted.

Theorem 2. *Let* $\sigma > 0$ *and* $\delta \in L^\infty(\mathbb{R})$*. Assume for the cut-off value* $U(\epsilon) \to \infty$ *and* $\epsilon U(\epsilon)^2\sqrt{\log(U(\epsilon))}\exp(T\sigma^2U(\epsilon)^2/2) \to 0$ *as* $\epsilon \to 0$*. Let* $w^1_\sigma, w^1_\gamma, w^1_\lambda, w^1_\mu : [0,1] \to \mathbb{R}$ *be Riemann-integrable, in* $L^\infty([0,1])$ *and continuous at one. For* $x \in \mathbb{R}$ *it holds with the notation* $U := U(\epsilon)$

$$\frac{1}{\epsilon e^{T\sigma^2U^2/2}}\begin{pmatrix} 2\int_0^1 \mathrm{Re}(\Delta\psi_\epsilon(Uu))w^1_\sigma(u)\mathrm{d}u - d\,w^1_\sigma(1)W_{\epsilon,U} \\ 2\int_0^1 \mathrm{Im}(\Delta\psi_\epsilon(Uu))w^1_\gamma(u)\mathrm{d}u - d\,w^1_\gamma(1)\;V_{\epsilon,U} \\ 2\int_0^1 \mathrm{Re}(\Delta\psi_\epsilon(Uu))w^1_\lambda(u)\mathrm{d}u - d\,w^1_\lambda(1)W_{\epsilon,U} \\ \mathcal{F}^{-1}\left[\Delta\psi_\epsilon(Uu)w^1_\mu(u)\right](Ux) - d\,w^1_\mu(1)\;Z_{\epsilon,U}(x)/(2\pi)\end{pmatrix} \xrightarrow{\mathbb{Q}} 0,$$

as $\epsilon \to 0$*, where* $Z_{\epsilon,U}(x) := \cos(Ux)W_{\epsilon,U} + \sin(Ux)V_{\epsilon,U}$*.*

The assumption $\mathcal{T} \in \mathcal{G}_s(R, \sigma_{\max})$ includes $\sigma \in [0, \sigma_{\max}]$. The condition $\epsilon U(\epsilon)^2\sqrt{\log(U(\epsilon))}\exp(T\sigma^2U(\epsilon)^2/2) \to 0$ is especially fulfilled if $U(\epsilon) \le \bar{\sigma}^{-1}(2\log(\epsilon^{-1})/T)^{1/2}$ for any $\bar{\sigma} > \sigma_{\max}$. For the estimation it suffices to know some upper bound $\sigma_{\max}$ of σ. The theorem shows that regardless whether one undersmoothes or not the stochastic errors converge with appropriate scaling to normal random variables. For the statement on asymptotic normality we have to undersmooth and further knowledge on the volatility is necessary.

In many situations the volatility σ is known or can be estimated easily. It can, for example, be determined from high frequency data of the underlying asset since the volatility is the same for equivalent measures. In the following we will assume either that the volatility σ is known as in Cont and Tankov [8] or that we have a sufficiently good estimator of the volatility. To control the remainder term we choose $U(\epsilon)$ such that

$\epsilon U(\epsilon)^2\sqrt{\log(U(\epsilon))}\exp(T\sigma^2 U(\epsilon)^2/2) \to 0$ as $\epsilon \to 0$. We also assume the undersmoothing condition $\epsilon U(\epsilon)^{s+1}\exp(T\sigma^2 U(\epsilon)^2/2) \to \infty$ as $\epsilon \to 0$. A smoothness parameter $s \geq 2$ is implicitly assumed so that both condition can be satisfied simultaneously. A possible choice of $U(\epsilon)$ is

$$U(\epsilon) := \sqrt{\frac{2}{T\sigma^2}\log\left(\frac{\epsilon^{-1}}{\log(\epsilon^{-1})^\alpha}\right)}, \tag{3.6}$$

where $\alpha \in (1, (s+1)/2)$. Then it holds

$$\epsilon U(\epsilon)^\beta \exp(T\sigma^2 U(\epsilon)^2/2) \to \begin{cases} \infty & \beta > 2\alpha \\ \left(\frac{2}{T\sigma^2}\right)^\alpha & \beta = 2\alpha \\ 0 & \beta < 2\alpha \end{cases}$$

as $\epsilon \to 0$. Especially the term diverges for $\beta = s+1$ and converges to zero for $\beta \in (2, 2\alpha)$ so that both conditions on $U(\epsilon)$ are fulfilled.

Next we state the joint asymptotic normality result for the severely ill-posed case of positive volatility.

Theorem 3. *Let $\sigma > 0$ and let the cut-off value $U(\epsilon)$ be chosen such that $\epsilon U(\epsilon)^2\sqrt{\log(U(\epsilon))}\exp(T\sigma^2 U(\epsilon)^2/2) \to 0$ and $\epsilon U(\epsilon)^{s+1}\exp(T\sigma^2 U(\epsilon)^2/2) \to \infty$ as $\epsilon \to 0$. Let $\delta \in L^\infty(\mathbb{R})$. It holds*

$$\frac{1}{\epsilon e^{T\sigma^2 U(\epsilon)^2/2}} \begin{pmatrix} U(\epsilon)^2 & \Delta\hat{\sigma}^2 & - & d\, w_\sigma^1(1) W_{\epsilon,U(\epsilon)} \\ U(\epsilon) & \Delta\hat{\gamma} & - & d\, w_\gamma^1(1) V_{\epsilon,U(\epsilon)} \\ & \Delta\hat{\lambda} & - & d\, w_\lambda^1(1) W_{\epsilon,U(\epsilon)} \\ U(\epsilon)^{-1} & \Delta\hat{\mu}(0) & - & d\, w_0(1) W_{\epsilon,U(\epsilon)}/(2\pi) \\ U(\epsilon)^{-1} & \Delta\hat{\mu}(x) & - & d\, w_\mu^1(1) Z_{\epsilon,U(\epsilon)}(x)/(2\pi) \end{pmatrix} \xrightarrow{\mathbb{Q}} 0,$$

as $\epsilon \to 0$, where $x \in \mathbb{R}\backslash\{0\}$, $Z_{\epsilon,U}(x) := \cos(Ux)W_{\epsilon,U} + \sin(Ux)V_{\epsilon,U}$ and d is given by (3.4).

3.2 Discussion of the results

Theorems 1 and 3 include the asymptotic distribution of $\hat{\sigma}^2$, which may be used for testing the hypotheses $H_0 : \sigma = \sigma_0$. If σ is known, we can set $\hat{\sigma}^2 = \sigma^2$. Then the statements of the theorems hold with w_σ^1 constant to zero. The estimation method can give negative values for $\hat{\sigma}^2$, $\hat{\lambda}$ and $\hat{\nu}(x)$. By a postprocessing step the estimated values can be corrected to be non-negative.

In Theorem 1 the noise level δ enters only locally into the asymptotic variance whereas in Theorems 2 and 3 the asymptotic variance depends on the L^2-norm of δ through the factor d. In fact for $\sigma = 0$ it is possible to estimate γ and λ directly from local properties of the option function $\mathcal{O}$ at γT as remarked in [2]. This local dependence on the noise level resembles some similarity to deconvolution, for instance, to the case of ordinary smooth

error density, where the density of the observations enters locally into the asymptotic variance [11, 23]. For the weight functions the local and global dependence is vice versa. In Theorem 1 the weight functions w^1_σ, w^1_γ, w^1_λ, w^1_0 and w^1_μ enter globally into the asymptotic variance while in Theorems 2 and 3 only the values of the weights functions at their endpoints appear in the asymptotic variance.

The asymptotic variance depends on the maturity. For positive volatility this dependence is through d in (3.4). The martingale condition is equivalent to $\sigma^2/2+\gamma-\lambda+\int_{-\infty}^{\infty} e^x \nu(x)\mathrm{d}x = 0$, especially it holds that $\lambda-\gamma-\sigma^2/2 \geq 0$ with equality if and only if $\lambda = 0$ that is in the Black-Scholes case. In the case of positive volatility σ the asymptotic variance grows exponentially as $T \to \infty$ if the jump intensity λ is positive and grows quadratic as $T \to 0$.

For $w^1_\sigma(1)$, $w^1_\gamma(1)$, $w^1_\lambda(1)$, $w^1_\mu(1) \in \mathbb{R}\backslash\{0\}$ Theorem 2 describes the asymptotic distribution of the leading stochastic error term of $\hat\sigma^2$, $\hat\gamma$, $\hat\lambda$ and $\hat\mu(x)$, $x \neq 0$, i.e., the other stochastic error terms are of smaller order. The variances in Theorems 2 and 3 converge by (3.5) and by the definition of $Z_{\epsilon,U}(x)$. If one only considers the stochastic errors of $\hat\sigma^2$, $\hat\gamma$, $\hat\lambda$ and $\hat\mu(0)$, then the covariances converge, too. But for $x \neq 0$ the covariance of the stochastic errors of $\hat\mu(x)$ and of $\hat\sigma^2$ does not converge. The same holds for the covariance of the stochastic errors of $\hat\mu(x)$ and $\hat\gamma$ as well as $\hat\mu(x)$ and $\hat\lambda$. The phenomenon that the covariances do not convergence comes from the fact that the stochastic error centers more and more at the cut-off frequency. The sequence of cut-off values has a crucial influence on the covariance. For estimators of the generalized distribution function of the Lévy density this is likely to lead to a similar dependence on the sequence of cut-off values as observed in [24] for deconvolution with a supersmooth error density.

4 Applications

4.1 Construction of confidence intervals and confidence sets

For $\sigma = 0$ we define confidence intervals

$$\begin{aligned}
I_{\gamma,\epsilon} &:= [\hat\gamma - \hat s_\gamma \epsilon U^{1/2} q_{\alpha/2}, && \hat\gamma + \hat s_\gamma \epsilon U^{1/2} q_{\alpha/2}],\\
I_{\lambda,\epsilon} &:= [\hat\lambda - \hat s_\lambda \epsilon U^{3/2} q_{\alpha/2}, && \hat\lambda + \hat s_\lambda \epsilon U^{3/2} q_{\alpha/2}],\\
I_{\mu(0),\epsilon} &:= [\hat\mu(0) - \hat s_{\mu(0)} \epsilon U^{5/2} q_{\alpha/2}, && \hat\mu(0) + \hat s_{\mu(0)} \epsilon U^{5/2} q_{\alpha/2}],\\
I_{\mu(x),\epsilon} &:= [\hat\mu(x) - \hat s_{\mu(x)} \epsilon U^{5/2} q_{\alpha/2}, && \hat\mu(x) + \hat s_{\mu(x)} \epsilon U^{5/2} q_{\alpha/2}],
\end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}\backslash\{0\}$, q_α denotes the $(1-\alpha)$-quantile of the standard normal distribution and

$$\begin{pmatrix} \hat{s}_\gamma \\ \hat{s}_\lambda \\ \hat{s}_{\mu(0)} \\ \hat{s}_{\mu(x)} \end{pmatrix} := \begin{pmatrix} \hat{s}(0)\left(\int_0^1 u^4 w_\gamma^1(u)^2 \mathrm{d}u\right)^{1/2} \\ \hat{s}(0)\left(\int_0^1 u^4 w_\lambda^1(u)^2 \mathrm{d}u\right)^{1/2} \\ \hat{s}(0)\left(\int_0^1 u^4 w_0(u)^2 \mathrm{d}u\right)^{1/2}/(2\pi) \\ \hat{s}(x)\left(\int_0^1 u^4 w_\mu^1(u)^2 \mathrm{d}u\right)^{1/2}/(2\pi) \end{pmatrix}$$

with $\hat{s}(x) := 2\sqrt{\pi}\delta(x+T\hat{\gamma})\exp(T(\hat{\lambda}-\hat{\gamma}))/T$. We fix some arbitrarily slowly decreasing function h with $h(u) \to 0$ as $|u| \to \infty$. We denote by $\mathcal{H}_s(R, \sigma_{\max})$ the subset of Lévy triplets in $\mathcal{G}_s(R, \sigma_{\max})$ that satisfy in addition

$$\|\mathcal{F}\mu\|_\infty \leq R, \quad |\mathcal{F}\mu(u)| \leq R\, h(u), \quad \forall u \in \mathbb{R}. \tag{4.2}$$

The additional conditions are used to extend the convergence in the theorems to uniform convergence, see Theorem 4.

Corollary. *Let $\sigma = 0$. On the assumptions of Theorem 1 and on the assumption that δ is positive and continuous*

$$\lim_{\epsilon\to 0} \inf_{\mathcal{T}\in\mathcal{H}_s(R,0)} \mathbb{Q}_{\mathcal{T}}(\rho \in I_{\rho,\epsilon}) = 1-\alpha$$

holds for the intervals (4.1) *and for all* $\rho \in \{\gamma, \lambda, \mu(x) | x \in \mathbb{R}\}$.

Remark. We can take the two parameters γ and λ and define $A_\epsilon := \{(\hat{\gamma} + \epsilon U^{1/2}\hat{s}_\gamma x, \hat{\lambda} + \epsilon U^{3/2}\hat{s}_\lambda y)^\top | x^2+y^2 \leq k_\alpha\}$, where k_α denotes the $(1-\alpha)$-quantile of the chi-square distribution χ_2^2 with two degrees of freedom. Then it holds

$$\lim_{\epsilon\to 0} \inf_{\mathcal{T}\in\mathcal{H}_s(R,0)} \mathbb{Q}_{\mathcal{T}}((\gamma,\lambda)^\top \in A_\epsilon) = 1-\alpha.$$

For $\sigma > 0$ we define confidence intervals

$$\begin{aligned} I_{\gamma,\epsilon} &:= [\hat{\gamma} - \hat{s}_\gamma \epsilon U^{-1} e^{T\sigma^2 U^2/2} q_{\alpha/2}, & \hat{\gamma} + \hat{s}_\gamma \epsilon U^{-1} e^{T\sigma^2 U^2/2} q_{\alpha/2}], \\ I_{\lambda,\epsilon} &:= [\hat{\lambda} - \hat{s}_\lambda \epsilon e^{T\sigma^2 U^2/2} q_{\alpha/2}, & \hat{\lambda} + \hat{s}_\lambda \epsilon e^{T\sigma^2 U^2/2} q_{\alpha/2}], \\ I_{\mu(0),\epsilon} &:= [\hat{\mu}(0) - \hat{s}_{\mu(0)} \epsilon U e^{T\sigma^2 U^2/2} q_{\alpha/2}, & \hat{\mu}(0) + \hat{s}_{\mu(0)} \epsilon U e^{T\sigma^2 U^2/2} q_{\alpha/2}], \\ I_{\mu(x),\epsilon} &:= [\hat{\mu}(x) - \hat{s}_\mu \epsilon U e^{T\sigma^2 U^2/2} q_{\alpha/2}, & \hat{\mu}(x) + \hat{s}_\mu \epsilon U e^{T\sigma^2 U^2/2} q_{\alpha/2}], \end{aligned} \tag{4.3}$$

where $x \in \mathbb{R}\backslash\{0\}$,

$$\begin{pmatrix} \hat{s}_\gamma \\ \hat{s}_\lambda \\ \hat{s}_{\mu(0)} \\ \hat{s}_\mu \end{pmatrix} := \frac{\sqrt{2}\|\delta\|_{L^2(\mathbb{R})}}{\exp(T(\sigma^2/2+\hat{\gamma}-\hat{\lambda}))T^2\sigma^2} \begin{pmatrix} |w_\gamma^1(1)| \\ |w_\lambda^1(1)| \\ |w_0(1)|/(2\pi) \\ |w_\mu^1(1)|/(2\pi) \end{pmatrix}$$

and q_α denotes the $(1-\alpha)$-quantile of the standard normal distribution.

Corollary. *Let $\sigma > 0$. On the assumptions of Theorem 3*

$$\lim_{\epsilon\to 0}\inf_{\mathcal{T}} \mathbb{Q}_{\mathcal{T}}(\rho \in I_{\rho,\epsilon}) = 1-\alpha$$

holds for the intervals (4.3) *and for all $\rho \in \{\gamma, \lambda, \mu(x)|x \in \mathbb{R}\}$, where the infimum is over all $\mathcal{T} \in \mathcal{H}_s(R, \sigma_{\max})$ with volatility σ.*

For $(\gamma,\lambda)^\top$ a uniform confidence set may be obtained similarly as in the case $\sigma = 0$. Since for $x \in \mathbb{R}\backslash\{0\}$ the covariance of $Z_{\epsilon,U(\epsilon)}(x)$ and $V_{\epsilon,U(\epsilon)}$ and the covariance of $Z_{\epsilon,U(\epsilon)}(x)$ and $W_{\epsilon,U(\epsilon)}$ do not converge, confidence sets for $(\gamma,\mu(x))^\top$ and $(\lambda,\mu(x))^\top$ have to be constructed differently. Let us illustrate how to proceed in this case by constructing a confidence set for $(\mu(x_1),\mu(x_2))^\top$, $x_1, x_2 \in \mathbb{R}\backslash\{0\}$. By Theorem 3 the convergence

$$\frac{1}{\epsilon e^{T\sigma^2 U(\epsilon)^2/2}}\left(\frac{1}{U(\epsilon)}\begin{pmatrix}\Delta\hat{\mu}(x_1)\\ \Delta\hat{\mu}(x_2)\end{pmatrix} - \frac{d\,w_\mu^1(1)}{2\pi}\begin{pmatrix}Z_{\epsilon,U(\epsilon)}(x_1)\\ Z_{\epsilon,U(\epsilon)}(x_2)\end{pmatrix}\right) \xrightarrow{\mathbb{Q}} 0$$

holds for $\epsilon \to 0$. We define

$$M_U := \begin{pmatrix}\cos(Ux_1) & \sin(Ux_1)\\ \cos(Ux_2) & \sin(Ux_2)\end{pmatrix},$$

and observe that the components of M_U^{-1} are bounded for U for which the absolute value of the determinant is bounded from below by some $c > 0$, i.e., $|\sin(U(x_2 - x_1))| \geq c$. For such $U(\epsilon)$

$$\frac{1}{\epsilon e^{T\sigma^2 U(\epsilon)^2/2}}\left(\frac{M_{U(\epsilon)}^{-1}}{U(\epsilon)}\begin{pmatrix}\Delta\hat{\mu}(x_1)\\ \Delta\hat{\mu}(x_2)\end{pmatrix} - \frac{d\,w_\mu^1(1)}{2\pi}\begin{pmatrix}W_{\epsilon,U(\epsilon)}(x_1)\\ V_{\epsilon,U(\epsilon)}(x_2)\end{pmatrix}\right) \xrightarrow{\mathbb{Q}} 0$$

holds for $\epsilon \to 0$. By Slutsky's lemma we obtain

$$\frac{1}{\hat{s}_\mu \epsilon U(\epsilon) e^{T\sigma^2 U(\epsilon)^2/2}} M_{U(\epsilon)}^{-1}\begin{pmatrix}\Delta\hat{\mu}(x_1)\\ \Delta\hat{\mu}(x_2)\end{pmatrix} \xrightarrow{d} \begin{pmatrix}W\\ V\end{pmatrix}$$

for $\epsilon \to 0$ such that $|\sin(U(\epsilon)(x_2 - x_1))| \geq c$. We define

$$C_\epsilon := \begin{pmatrix}\hat{\mu}(x_1)\\ \hat{\mu}(x_2)\end{pmatrix} + \begin{pmatrix}\cos(U(\epsilon)x_1) & \sin(U(\epsilon)x_1)\\ \cos(U(\epsilon)x_2) & \sin(U(\epsilon)x_2)\end{pmatrix} B_\epsilon,$$

where $B_\epsilon := \{\hat{s}_\mu \epsilon U(\epsilon) e^{T\sigma^2 U(\epsilon)^2/2}(x,y)^\top |\, x^2 + y^2 \leq k_\alpha\}$ and k_α denotes the $(1-\alpha)$-quantile of the chi-square distribution χ_2^2 with two degrees of freedom. Then

$$\lim_{\substack{|\sin(U(\epsilon)(x_2-x_1))|\geq c\\ \epsilon\to 0}} \mathbb{Q}_{\mathcal{T}}((\mu(x_1),\mu(x_2))^\top \in C_\epsilon) = 1-\alpha$$

holds for all $\mathcal{T} \in \mathcal{G}_s(R, \sigma_{\max}) \cap \{\sigma > 0\}$.

4.2 Inference on the volatility

The results on asymptotic normality allow to test for the value of the volatility. Let $\sigma_0 \in [0, \sigma_{\max}]$ and define the hypotheses $H_0 : \mathcal{T} \in \mathcal{H}_s(R, \sigma_{\max}) \cap \{\sigma = \sigma_0\}$ and the alternative $H_1 : \mathcal{T} \in \mathcal{H}_s(R, \sigma_{\max}) \cap \{|\sigma - \sigma_0| \geq \tau\}$ for some $\tau > 0$. For $\alpha \in (0, 1/2]$ there is a test, which reaches asymptotically the level α, i.e., $\lim_{\epsilon\to 0} \sup_{\mathcal{T}} \mathbb{E}_{\mathcal{T}}[\varphi_{\sigma_0}] = \alpha$, where the supremum is over all $\mathcal{T}$ in the hypotheses H_0. Moreover, the error of the second kind vanishes asymptotically, i.e., $\lim_{\epsilon\to 0} \sup_{\mathcal{T}} \mathbb{E}_{\mathcal{T}}[1 - \varphi_{\sigma_0}] = 0$, where the supremum is over all $\mathcal{T}$ in the alternative H_1. This family of tests can be used to construct a confidence set for σ. The precise construction of the test and of the confidence set is given in Section 7.

4.3 A numerical example

We consider the *Merton* jump diffusion model [18], where the jump density is specified by

$$\nu(x) = \frac{\lambda}{\sqrt{2\pi}\, v} \exp\left(-\frac{(x-\eta)^2}{2v^2}\right), \quad x \in \mathbb{R},$$

with parameters $\sigma, \lambda \geq 0$, $v > 0$, $\eta \in \mathbb{R}$ and where $\gamma \in \mathbb{R}$ is determined by the martingale condition. We simulate data with the parameters $\sigma = 0.1$, $\lambda = 5$, $\eta = -0.1$, $v = 0.2$, which implies $\gamma = 0.379$. The interest rate is taken to be $r = 0.06$. We observe prices of $n = 100$ European options with maturity $T = 0.25$. The strike prices are obtained from sampling the data points (x_j) from a centered normal distribution with variance $1/2$, so that more strike prices are sampled at the money than in or out of the money. The observation error is chosen to be a centered normal distribution with variance $\delta^2 \mathcal{O}(x_j)^2$, $\delta = 0.01$. Belomestny and Reiß [3] describe the implementation of the estimation method in detail.

We interpolate the corresponding European call prices linearly. The weight functions are chosen as in [3] with smoothness parameter $s = 2$, there denoted by r. In the simulations the confidence intervals based on the asymptotic distribution turn out to be to conservative. The asymptotic confidence intervals are based on the asymptotic variance of the linearized stochastic errors that is the stochastic errors, where the linearization (3.2) is used. Instead of taking the asymptotic variance of the linearized stochastic errors we derive confidence intervals from the finite sample variance (6.17) of the linearized stochastic errors. In the finite sample variance we substitute σ, γ, λ, μ and ν by their respective estimators. This yields feasible confidence intervals. We bound the oracle cut-off value at 30 and perform 1000 Monte Carlo iterations. The coverage probabilities of 95% confidence intervals for σ^2, γ and λ are approximately 0.99, 0.95 and 0.97, respectively. We see that the that the coverage probabilities are close to the prescribed confidence level.

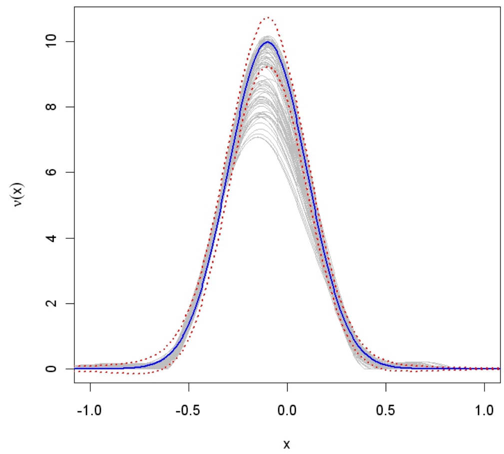

Figure 1: True Lévy density (blue, solid) with pointwise confidence intervals (red, dashed) and 100 estimated Lévy densities (grey).

Figure 1 illustrates the true Lévy density and the pointwise 95% confidence intervals based on the finite sample variance. At zero there is a negative bias since the peak is smoothed out. Furthermore, the theory predicts a larger confidence interval at zero involving the effective weight function w_0. This phenomenon extends in the simulations to a neighborhood of zero. The precise construction of the confidence intervals and more simulation results are contained in [20].

5 Conclusion

We have shown asymptotic normality in a nonparametric calibration method for exponential Lévy models. These results were used to derive confidence intervals and confidence sets as well as to construct a test on the value of the volatility. We have seen in a numerical example that confidence intervals based on finite sample variance perform well in terms of coverage probabilities. The confidence intervals extend the calibration method beyond a pure point estimate and enable an assessment of the calibration error. Although parametric models might be fitted better, the parametric approach is always exposed to the risk of model misspecification and the obtained confidence results should be used for a goodness-of-fit test.

The estimation method and the asymptotic normality results may be adapted to other models as long as there is an equation relating the option function to the characteristic function and the parameters of interest appear

in the characteristic function. The constructed confidence intervals and sets may be used to quantify the errors in pricing, hedging and risk management.

6 Proof of the asymptotic normality

First, we write $\Delta\hat{\gamma}$, $\Delta\hat{\lambda}$ and $\Delta\hat{\mu}(x)$ similarly as in (2.8) for $\Delta\hat{\sigma}^2$:

$$\Delta\hat{\gamma} = -\Delta\hat{\sigma}^2 + 2U^{-1}\int_0^1 \mathrm{Im}(\mathcal{F}\mu(Uu))w_\gamma^1(u)\mathrm{d}u \qquad (6.1)$$
$$+ 2U^{-1}\int_0^1 \mathrm{Im}(\Delta\psi_\epsilon(Uu))w_\gamma^1(u)\mathrm{d}u,$$
$$\Delta\hat{\lambda} = \Delta\hat{\sigma}^2/2 + \Delta\hat{\gamma} - 2\int_0^1 \mathrm{Re}(\mathcal{F}\mu(Uu))w_\lambda^1(u)\mathrm{d}u \qquad (6.2)$$
$$- 2\int_0^1 \mathrm{Re}(\Delta\psi_\epsilon(Uu))w_\lambda^1(u)\mathrm{d}u,$$
$$\begin{aligned}\Delta\hat{\mu}(x) = {} & U\mathcal{F}^{-1}\left[\Delta\psi_\epsilon(Uu)w_\mu^1(u)\right](Ux) \qquad (6.3)\\ & + \frac{\Delta\hat{\sigma}^2}{2}U\mathcal{F}^{-1}\left[(Uu-i)^2w_\mu^1(u)\right](Ux)\\ & - i\Delta\hat{\gamma}U\mathcal{F}^{-1}\left[(Uu-i)w_\mu^1(u)\right](Ux)\\ & + \Delta\hat{\lambda}U\mathcal{F}^{-1}\left[w_\mu^1(u)\right](Ux)\\ & - U\mathcal{F}^{-1}\left[(1-w_\mu^1(u))\mathcal{F}\mu(Uu)\right](Ux).\end{aligned}$$

In (6.1) we can substitute $\Delta\hat{\sigma}^2$ using (2.8) and obtain two error terms involving $\mathcal{F}\mu$ and two error terms involving $\Delta\psi_\epsilon$. By similar substitutions in (6.2) and (6.3) we see that all error terms either involve $\mathcal{F}\mu$ or $\Delta\psi_\epsilon$, which we will call approximation errors and stochastic errors, respectively.

We will now state the conditions more precisely on which the regression model (2.1) and the Gaussian white noise model (3.1) are equivalent. We restrict the Gaussian white noise model to a growing sequence of intervals $[\alpha_n, \beta_n]$ and assume as a simplification that the observations in the regression model are equidistant on these intervals with mesh size Δ_n. We assume that $(\beta_n - \alpha_n)\Delta_n \to 0$. We suppose $\delta^2 > 0$ to be an absolutely continuous function and $\frac{\partial}{\partial x}\log\delta(x) \leq C$ to hold for some $C < \infty$. The functions $\mathcal{O}$ are uniformly bounded $\mathcal{O}(x) = S^{-1}\mathcal{C}(x,T) - (1-e^x)^+ \leq 1$ and uniformly Lipschitz $|\mathcal{O}'(x)| = |\int_{-\infty}^x \mathcal{O}''(x)\mathrm{d}x - \mathbf{1}_{\{x>0\}} + e^{(\gamma-\lambda)T}\mathbf{1}_{\{x>\gamma T,\sigma=0\}}| \leq 4 + e^{RT}$, where we used Proposition 2.1 in [2] and $|\gamma| \leq R$. These properties of $\mathcal{O}$ are used to apply Corollary 4.2 in [4]. Then ϵ corresponds to $(\beta_n - \alpha_n)/\sqrt{n}$, especially for logarithmically growing intervals one loses only a logarithmic factor.

6.1 Proof of Proposition 1

First we define $X(u) := \int_{-\infty}^{\infty} e^{iux}\delta(x)\mathrm{d}W(x)$. We assumed that there is an $p > 1$ such that $\int_{-\infty}^{\infty}(1+|x|)^p\delta(x)^2\mathrm{d}x < \infty$. It is shown in [19] that there exists a number $c > 0$ such that $\sqrt{\mathbb{E}[|X(u) - X(v)|^2]} \leq c|u-v|^\alpha$ for all $u, v \in \mathbb{R}$ with $\alpha := \min(p/2, 1) \in (1/2, 1]$. Denote by $N_\rho(I, r)$ the covering number, that is the minimum number of closed balls of radius r in the metric ρ with centers in I that cover I. We define $\rho(u,v) := c|u-v|^\alpha$ and $d(u,v) := \sqrt{\mathbb{E}[|X(u) - X(v)|^2]}$. A ball of radius r in the metric ρ covers an interval of length $2(r/c)^{1/\alpha}$. Thus, it holds

$$N_\rho([-U,U], r) = \left\lceil U\,(c/r)^{1/\alpha}\right\rceil .$$

The radius of the smallest ball with center in $[-U,U]$ that contains $[-U,U]$ is cU^α with respect to the metric ρ. There exists $D < \infty$ such that $d(u,v) \leq D$ for all $u, v \in \mathbb{R}$. For U large enough such that $cU^\alpha \geq D$ we have the entropy bound

$$\begin{aligned} J([-U,U], d) &= \int_0^\infty (\log(N_d([-U,U], r)))^{1/2}\,\mathrm{d}r \qquad (6.4)\\ &= \int_0^D (\log(N_d([-U,U], r)))^{1/2}\,\mathrm{d}r \\ &\leq \int_0^D (\log(N_\rho([-U,U], r)))^{1/2}\,\mathrm{d}r \\ &\leq \int_0^D \left(\log\left(2U\,(c/r)^{1/\alpha}\right)\right)^{1/2}\mathrm{d}r \\ &\leq \int_0^D \left(\log\left((2^\alpha U^\alpha c/r)^{1/\alpha}\right)\right)^{1/2}\mathrm{d}r \\ &\leq \alpha^{-1/2}\int_0^D (\log(2^\alpha U^\alpha c/r))^{1/2}\,\mathrm{d}r, \end{aligned}$$

here we substitute $r = 2^\alpha U^\alpha cs$,

$$\leq \alpha^{-1/2} 2^\alpha U^\alpha c \int_0^{D/(2^\alpha U^\alpha c)} (\log(1/s))^{1/2}\,\mathrm{d}s. \qquad (6.5)$$

This integral is solved by

$$\int_0^x \sqrt{\log y^{-1}}\mathrm{d}y = \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\mathrm{Erf}(\sqrt{\log x^{-1}}) + x\sqrt{\log x^{-1}},$$

where $\mathrm{Erf}(y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{-t^2}\mathrm{d}t$. For all $y > 0$ the estimate $1 - \mathrm{Erf}(y) < \exp(-y^2)/(\sqrt{\pi}y)$ holds. For each $\alpha > 0$ this yields $\tilde{c} > 0$ such that for all $x \in]0, 1/2^\alpha[$

$$\int_0^x \sqrt{\log y^{-1}}\mathrm{d}y \leq \tilde{c}x\sqrt{\log x^{-1}}.$$

Thus, (6.5) can be bounded by

$$(\alpha^{-1/2}\tilde{c}D)\sqrt{\log(2^\alpha U^\alpha c/D)} \lesssim \sqrt{\log(U)}$$

as $U \to \infty$. Thus, $\sqrt{\log(U)}$ is an asymptotic upper bound of the entropy integral (6.4). We apply Dudley's theorem [15, p. 219] to the real part of X. For all $q \geq 1$ this yields a continuous version X' of $\mathrm{Re}(X)$ with

$$\mathbb{E}\left[\sup_{u\in[-U,U]} |X'(u)|^q\right] \lesssim (\log(U))^{q/2} \tag{6.6}$$

as $U \to \infty$. Since X' and $\mathrm{Re}(X)$ are both continuous they are indistinguishable and (6.6) holds for $\mathrm{Re}(X)$ likewise. We obtain analogously for all $q \geq 1$

$$\mathbb{E}\left[\sup_{u\in[-U,U]} |\mathrm{Im}(X(u))|^q\right] \lesssim (\log(U))^{q/2}$$

as $U \to \infty$. We estimate from above for all $q \geq 1$

$$\begin{aligned}
&\mathbb{E}\left[\sup_{u\in[-1,1]} |\mathcal{L}_{\epsilon,U}(u)|^q\right] \\
&\leq \sup_{u\in[-U,U]} \left|\frac{\epsilon iu(1+iu)}{T(1+iu(1+iu)\mathcal{F}\mathcal{O}(u))}\right|^q \mathbb{E}\left[\sup_{u\in[-U,U]} |X(u)|^q\right] \\
&\leq \left(\frac{\epsilon U\sqrt{1+U^2}}{T\exp(T(-\sigma^2U^2/2+\sigma^2/2+\gamma-\lambda-\|\mathcal{F}\mu\|_\infty))}\right)^q \mathbb{E}\left[\sup_{u\in[-U,U]} |X(u)|^q\right] \\
&\lesssim \left(\epsilon U^2\sqrt{\log(U)}\exp(T\sigma^2U^2/2)\right)^q
\end{aligned} \tag{6.7}$$

as $U \to \infty$.

6.2 The linearized stochastic errors

The linearized stochastic errors are of the form $\int_0^1 f_j(u)\mathcal{L}_{\epsilon,U}(u)\mathrm{d}u$, where f_j with $j = 1,\ldots,n$ are Riemann-integrable function in $L^\infty([0,1])$. Next we will show that these are jointly normal distributed. Almost surely $\mathcal{L}_{\epsilon,U}$ is continuous. Thus, almost surely the $f_j\mathcal{L}_{\epsilon,U}$ are Riemann-integrable and almost surely

$$\frac{1}{m}\sum_{k=1}^m f_j(k/m)\mathcal{L}_{\epsilon,U}(k/m) \to \int_0^1 f_j(u)\mathcal{L}_{\epsilon,U}(u)\mathrm{d}u$$

as $m \to \infty$. Let $C > 0$ be such that $\|f_j\|_\infty \leq C$ for all $j = 1,\ldots,n$. For each m the n sums are joint, centered normal random variables. For $m \to \infty$ the

covariance matrix converges by the dominated convergence theorem with the dominating function $C^2 \sup_{u\in[0,1]} |\mathcal{L}_{\epsilon,U}(u)|^2$, where $\sup_{u\in[0,1]} |\mathcal{L}_{\epsilon,U}(u)|^2$ is an integrable random variable by Proposition 1. Thus, the characteristic function converges pointwise. By Lévy's continuity theorem this shows that the sums convergence jointly in distribution to normal random variables. So $\int_0^1 f_j(u)\mathcal{L}_{\epsilon,U}(u)\mathrm{d}u$ are jointly normal distributed.

For a fixed cut-off value U the linearized stochastic errors are jointly normal distributed. So the natural question is whether the appropriately scaled covariance matrix converges for $U \to \infty$.

Let $w_j, w_k : [0,1] \to \mathbb{R}$ be Riemann-integrable functions in $L^\infty([0,1])$. It holds

$$
\begin{aligned}
&T \int_0^1 w_j(u)\mathcal{L}_{\epsilon,U}(u)e^{-iUux_j}\mathrm{d}u \\
&= \epsilon U^2 e^{-T(\sigma^2/2+\gamma-\lambda)} \int_0^1 f_U(u) \int_{-\infty}^{\infty} e^{iUu(x-\theta_j)+T\sigma^2U^2u^2/2}\delta(x)\mathrm{d}W(x)\mathrm{d}u,
\end{aligned}
$$

where

$$
f_U(u) := \frac{w_j(u)(-u^2+iu/U)}{\exp(T\mathcal{F}\mu(Uu))} \tag{6.8}
$$

and $\theta_j := x_j + T\sigma^2 + T\gamma$. We define analogously

$$
g_U(u) := \frac{w_k(u)(-u^2+iu/U)}{\exp(T\mathcal{F}\mu(Uu))} \tag{6.9}
$$

and $\theta_k := x_k + T\sigma^2 + T\gamma$. We extend f_U and g_U by zero outside the interval $[0,1]$.

Since $\mathbb{E}\left[\sup_{u\in[-1,1]} |\mathcal{L}_{\epsilon,U}(u)|^2\right] < \infty$ we may apply Fubini's theorem and then we apply the Itô isometry to obtain

$$
\begin{aligned}
&T^2 e^{2T(\sigma^2/2+\gamma-\lambda)} \mathbb{E}\left[\int_0^1 w_j(u)\mathcal{L}_{\epsilon,U}(u)e^{-iUux_j}\mathrm{d}u \overline{\int_0^1 w_k(v)\mathcal{L}_{\epsilon,U}(v)e^{-iUvx_k}\mathrm{d}v}\right] \\
&= \epsilon^2 U^4 \int_0^1 \int_0^1 \int_{-\infty}^{\infty} f_U(u)e^{iUu(x-\theta_j)+T\sigma^2U^2u^2/2} \\
&\quad \overline{g_U(v)e^{iUv(x-\theta_k)+T\sigma^2U^2v^2/2}}\delta(x)^2\mathrm{d}x\mathrm{d}u\mathrm{d}v.
\end{aligned} \tag{6.10}
$$

To separate real and imaginary part we will also need

$$
\begin{aligned}
&T^2 e^{2T(\sigma^2/2+\gamma-\lambda)} \mathbb{E}\left[\int_0^1 w_j(u)\mathcal{L}_{\epsilon,U}(u)e^{-iUux_j}\mathrm{d}u \int_0^1 w_k(v)\mathcal{L}_{\epsilon,U}(v)e^{-iUvx_k}\mathrm{d}v\right] \\
&= \epsilon^2 U^4 \int_0^1 \int_0^1 \int_{-\infty}^{\infty} f_U(u)e^{iUu(x-\theta_j)+T\sigma^2U^2u^2/2} \\
&\quad g_U(v)e^{iUv(x-\theta_k)+T\sigma^2U^2v^2/2}\delta(x)^2\mathrm{d}x\mathrm{d}u\mathrm{d}v.
\end{aligned} \tag{6.11}
$$

Lemma 1. *Let $\sigma = 0$. For $j = 1, \dots, n$ let $x_j \in \mathbb{R}$ and let $w_j : [0,1] \to \mathbb{R}$ be Riemann-integrable functions in $L^\infty([0,1])$. Let δ be continuous at $x_1 + T\gamma$, $x_2 + T\gamma, \dots, x_n + T\gamma$ and let $\mathcal{F}\delta^2 \in L^1(\mathbb{R})$. Let $W_{x_1}, \dots, W_{x_n}, V_{x_1}, \dots, V_{x_n}$ be Brownian motions. If $x_j = x_k$ let $W_{x_j} = W_{x_k}$ and $V_{x_j} = V_{x_k}$ otherwise let the Brownian motions be distinct. Let the set $\{W_{x_1}, \dots, W_{x_n}, V_{x_1}, \dots, V_{x_n}\}$ consist of independent Brownian motions. Then*

$$\frac{1}{\epsilon U^{3/2}} \int_0^1 w_j(u) \mathcal{L}_{\epsilon,U}(u) e^{-iUux_j} \mathrm{d}u$$

converge jointly in distribution to

$$\frac{\sqrt{\pi}\delta(x_j + T\gamma)}{T \exp(T(\gamma - \lambda))} \left(\int_0^1 u^2 w_j(u) \mathrm{d}W_{x_j}(u) + i \int_0^1 u^2 w_j(u) \mathrm{d}V_{x_j}(u) \right)$$

as $U \to \infty$.

Proof. We will first consider the case $x_j = x_k$. We have seen that

$$\begin{aligned} &T^2 e^{2T(\gamma-\lambda)} \mathbb{E}\left[\int_0^1 w_j(u) \mathcal{L}_{\epsilon,U}(u) e^{-iUux_j} \mathrm{d}u \overline{\int_0^1 w_k(v) \mathcal{L}_{\epsilon,U}(v) e^{-iUvx_j} \mathrm{d}v} \right] \\ &= \epsilon^2 U^4 \int_{-\infty}^{\infty} \mathcal{F} f_U(U(x - \theta_j)) \overline{\mathcal{F} g_U(U(x - \theta_j))} \delta(x)^2 \mathrm{d}x, \end{aligned}$$

where f_U and g_U are defined as in (6.8) and (6.9), respectively, and $\theta_j = x_j + T\gamma$,

$$= \epsilon^2 U^3 \int_{-\infty}^{\infty} \mathcal{F} f_U(y) \overline{\mathcal{F} g_U(y) \delta(y/U + \theta_j)^2} \mathrm{d}y.$$

We notice that $\mathcal{F}\delta^2 \in L^1(\mathbb{R})$ implies $\delta^2 \in L^\infty(\mathbb{R})$ and we obtain by the Plancherel identity

$$= 2\pi\epsilon^2 U^3 \int_0^1 f_U(u) \overline{(g_U(v) * \mathcal{F}^{-1}(\delta(y/U + \theta_j)^2)(v))(u)} \mathrm{d}u, \tag{6.12}$$

since the support of f_U is $[0,1]$. Because we are only interested in the limit $U \to \infty$ we may assume $U \geq 1$. By the Riemann-Lebesgue lemma $\mathcal{F}\mu(u)$ tends to zero as $|u| \to \infty$. The factor $f_U(u)$ converges for each $u \in [0,1]$ to $-u^2 w_j(u)$ as $U \to \infty$ and the functions are dominated by a constant independent of U. In order to apply dominated convergence it suffices that the second factor is dominated by a constant independent of U and converges stochastically with respect to the Lebesgue measure on $\mathbb{R}$.

$$\begin{aligned} &(g_U(v) * \mathcal{F}^{-1}(\delta(y/U + \theta_j)^2)(v))(u) \\ &= \int_{-\infty}^{\infty} g_U(u - v) \mathcal{F}^{-1}(\delta(y + \theta_j)^2)(Uv) U \mathrm{d}v \end{aligned}$$

By assumption $\mathcal{F}\delta^2$ lies in $L^1(\mathbb{R})$ and so does $\mathcal{F}^{-1}(\delta(y+\theta_j)^2)$. A dominating constant is $\sqrt{2}\|w_k\|_\infty \exp(T\|\mathcal{F}\mu\|_\infty)\,\|\mathcal{F}^{-1}(\delta(y+\theta_j)^2)\|_{L^1(\mathbb{R})}$. It holds

$$\begin{aligned}\int_{-\infty}^{\infty} \mathcal{F}^{-1}(\delta(y+\theta_j)^2)(Uv)U\mathrm{d}v &= \int_{-\infty}^{\infty} \mathcal{F}^{-1}(\delta(y+\theta_j)^2)(v)\mathrm{d}v \\ &= \mathcal{F}\mathcal{F}^{-1}(\delta(y+\theta_j)^2)(0) = \delta(\theta_j)^2. \end{aligned} \tag{6.13}$$

$\delta_U(v) := \mathcal{F}^{-1}(\delta(y+\theta_j)^2)(Uv)U$ is the multiple of what is called approximate identity or nascent delta function. The basic theorem on approximate identities states that $h * \delta_n$ converges to h in $L^1(\mathbb{R})$ as $n \to \infty$ for $h \in L^1(\mathbb{R})$. Thus, $(-v^2 w_k(v)\mathbf{1}_{[0,1]}(v)) * \delta_U(v)(u)$ converges to $-u^2 w_k(u)\mathbf{1}_{[0,1]}(u)\delta(\theta_j)^2$ for $U \to \infty$ in $L^1(\mathbb{R})$ [13, p. 28] and in particular stochastically. If $u \neq 0$, then there is a neighborhood of u where $g_U(u) + u^2 w_k(u)\mathbf{1}_{[0,1]}(u)$ converges uniformly to zero. $(g_U(v) + v^2 w_k(v)\mathbf{1}_{[0,1]}(v)) * \delta_U(v)(u)$ converges almost surely and in particular stochastically to zero. Therefore, $g_U(v) * \delta_U(v)(u)$ converges to $-u^2 w_k(u)\mathbf{1}_{[0,1]}(u)\delta(\theta_j)^2$ stochastically with respect to the Lebesgue measure on $\mathbb{R}$.

We obtain under the limit $U \to \infty$ by the dominated convergence theorem

$$\begin{aligned}&\lim_{U\to\infty} \frac{1}{\epsilon^2 U^3}\mathbb{E}\left[\int_0^1 w_j(u)\mathcal{L}_{\epsilon,U}(u)e^{-iUux_j}\mathrm{d}u \overline{\int_0^1 w_k(v)\mathcal{L}_{\epsilon,U}(v)e^{-iUvx_j}\mathrm{d}v}\right] \\ &= \frac{2\pi\delta(\theta_j)^2}{T^2\exp(2T(\gamma-\lambda))}\int_{-\infty}^{\infty}\left(-u^2 w_j(u)\mathbf{1}_{[0,1]}(u)\right)\left(-u^2 w_k(u)\mathbf{1}_{[0,1]}(u)\right)\mathrm{d}u \\ &= \frac{2\pi\delta(\theta_j)^2}{T^2\exp(2T(\gamma-\lambda))}\int_0^1 u^4 w_j(u)w_k(u)\mathrm{d}u. \end{aligned} \tag{6.14}$$

Without taking the complex conjugate in (6.12) we obtain

$$\begin{aligned}&T^2 e^{2T(\gamma-\lambda)}\mathbb{E}\left[\int_0^1 w_j(u)\mathcal{L}_{\epsilon,U}(u)e^{-iUux_j}\mathrm{d}u \int_0^1 w_k(v)\mathcal{L}_{\epsilon,U}(v)e^{-iUvx_j}\mathrm{d}v\right] \\ &= 2\pi\int_{-\infty}^{\infty} f_U(u)\overline{(\overline{g_U(-v)} * \mathcal{F}^{-1}(\delta(y/U+\theta_j)^2)(v))(u)}\mathrm{d}u.\end{aligned}$$

The same argumentation as before leads to

$$\begin{aligned}&\lim_{U\to\infty}\mathbb{E}\left[\int_0^1 w_j(u)\mathcal{L}_{\epsilon,U}(u)e^{-iUux_j}\mathrm{d}u \int_0^1 w_k(v)\mathcal{L}_{\epsilon,U}(v)e^{-iUvx_j}\mathrm{d}v\right] \\ &= \frac{2\pi\delta(\theta_j)^2}{T^2\exp(2T(\gamma-\lambda))}\int_{-\infty}^{\infty}\left(-u^2 w_j(u)\mathbf{1}_{[0,1]}(u)\right)\left(-u^2 w_k(-u)\mathbf{1}_{[0,1]}(-u)\right)\mathrm{d}u \\ &= 0. \end{aligned} \tag{6.15}$$

We combine (6.14) and (6.15) to obtain

$$\begin{aligned}
&\lim_{U\to\infty}\frac{1}{\epsilon^2U^3}\mathbb{E}\left[\operatorname{Re}\int_0^1\frac{w_j(u)\mathcal{L}_{\epsilon,U}(u)}{\exp(iUux_j)}\mathrm{d}u\operatorname{Re}\int_0^1\frac{w_k(u)\mathcal{L}_{\epsilon,U}(u)}{\exp(iUux_j)}\mathrm{d}v\right]\\
&=\lim_{U\to\infty}\frac{1}{\epsilon^2U^3}\mathbb{E}\left[\operatorname{Im}\int_0^1\frac{w_j(u)\mathcal{L}_{\epsilon,U}(u)}{\exp(iUux_j)}\mathrm{d}u\operatorname{Im}\int_0^1\frac{w_k(u)\mathcal{L}_{\epsilon,U}(u)}{\exp(iUux_j)}\mathrm{d}v\right]\\
&=\frac{\pi\delta(\theta_j)^2}{T^2\exp(2T(\gamma-\lambda))}\int_0^1 u^4w_j(u)w_k(u)\mathrm{d}u.
\end{aligned}$$

From (6.14) and (6.15) it also follows that the covariance between real and imaginary part vanishes asymptotically.

In the case $x_j\neq x_k$ we have to show that the covariance vanishes asymptotically. Without loss of generality we assume $x_j<x_k$. We define $\theta:=(\theta_j+\theta_k)/2$. Then $\theta_j<\theta<\theta_k$.

$$\begin{aligned}
&\frac{T^2e^{2T(\gamma-\lambda)}}{\epsilon^2U^3}\mathbb{E}\left[\int_0^1 w_j(u)\mathcal{L}_{\epsilon,U}(u)e^{-iUux_j}\mathrm{d}u\overline{\int_0^1 w_k(v)\mathcal{L}_{\epsilon,U}(v)e^{-iUvx_k}\mathrm{d}v}\right]\\
&=\int_{-\infty}^{\infty}\mathcal{F}f_U(U(x-\theta_j))\overline{\mathcal{F}g_U(U(x-\theta_k))}\delta(x)^2U\mathrm{d}x\\
&=\int_{-\infty}^{\infty}\mathcal{F}f_U(y+U(\theta-\theta_j))\overline{\mathcal{F}g_U(y+U(\theta-\theta_k))}\delta(y/U+\theta)^2\mathrm{d}y
\end{aligned}$$

By the Plancherel identity and by the dominated convergence theorem

$$\mathcal{F}f_U\to\mathcal{F}\left(-u^2w_j(u)\right)\quad\text{and}\quad\mathcal{F}g_U\to\mathcal{F}\left(-u^2w_k(u)\right)$$

in $L^2(\mathbb{R})$ for $U\to\infty$ and especially the $L^2(\mathbb{R})$ norms converge. From the assumption $\mathcal{F}\delta^2\in L^1(\mathbb{R})$ follows that $\delta^2\in L^\infty(\mathbb{R})$. By the Cauchy-Schwarz inequality

$$\begin{aligned}
&\lim_{U\to\infty}\left|\int_{-\infty}^0\mathcal{F}f_U(y+U(\theta-\theta_j))\overline{\mathcal{F}g_U(y+U(\theta-\theta_k))}\delta(y/U+\theta)^2\mathrm{d}y\right|\\
&\leq\lim_{U\to\infty}\|\delta\|_\infty^2\,\|\mathcal{F}f_U\|_{L^2(\mathbb{R})}\left(\int_{-\infty}^{U(\theta-\theta_k)}|\mathcal{F}g_U(y)|^2\,\mathrm{d}y\right)^{1/2}=0.
\end{aligned}$$

A similar calculation shows that the integral over $(0,\infty)$ converges to zero and consequently

$$\lim_{U\to\infty}\frac{T^2e^{2T(\gamma-\lambda)}}{\epsilon^2U^3}\mathbb{E}\left[\int_0^1\frac{w_j(u)\mathcal{L}_{\epsilon,U}(u)}{\exp(iUux_j)}\mathrm{d}u\overline{\int_0^1\frac{w_k(v)\mathcal{L}_{\epsilon,U}(v)}{\exp(iUvx_k)}\mathrm{d}v}\right]=0.$$

The same way follows

$$\lim_{U\to\infty}\frac{T^2e^{2T(\gamma-\lambda)}}{\epsilon^2U^3}\mathbb{E}\left[\int_0^1\frac{w_j(u)\mathcal{L}_{\epsilon,U}(u)}{\exp(iUux_j)}\mathrm{d}u\int_0^1\frac{w_k(v)\mathcal{L}_{\epsilon,U}(v)}{\exp(iUvx_k)}\mathrm{d}v\right]=0.$$

The $1/(\epsilon U^{3/2})\int_0^1 w_j(u)\mathcal{L}_{\epsilon,U}(u)e^{-iUux_j}\mathrm{d}u$ are centered normal random variables and their covariance matrix converges to the covariance matrix of the claimed limit. Thus, the characteristic function converges pointwise. By Lévy's continuity theorem this shows the convergence in distribution. $\square$

Lemma 2. *Let $\sigma > 0$ and $\delta \in L^\infty(\mathbb{R})$. Let $w_U, \tilde{w}_U \in L^\infty([0,1],\mathbb{C})$ be Riemann-integrable and let there be a constant $C > 0$ such that for all $U \geq 1$ $\|w_U\|_\infty, \|\tilde{w}_U\|_\infty \leq C$. Let there be $a, \tilde{a} : [1,\infty) \to \mathbb{C}$ such that the condition*

$$\lim_{\delta\to 0}\sup_{U\geq 1}\sup_{u\in[1-\delta/U,1]}|w_U(u) - a(U)| = 0 \tag{6.16}$$

and the corresponding condition for $\tilde{w}_U$ and $\tilde{a}$ hold. Then

$$\lim_{U\to\infty}\frac{1}{\epsilon^2\exp(T\sigma^2U^2)}\mathbb{E}\left[\int_0^1 w_U(u)\mathcal{L}_{\epsilon,U}(u)\mathrm{d}u\int_0^1 \tilde{w}_U(v)\mathcal{L}_{\epsilon,U}(v)\mathrm{d}v\right] = 0,$$

$$\lim_{U\to\infty}\left(\frac{1}{\epsilon^2\exp(T\sigma^2U^2)}\mathbb{E}\left[\int_0^1 w_U(u)\mathcal{L}_{\epsilon,U}(u)\mathrm{d}u\overline{\int_0^1 \tilde{w}_U(v)\mathcal{L}_{\epsilon,U}(v)\mathrm{d}v}\right]\right.$$
$$\left. - \frac{a(U)\overline{\tilde{a}(U)}\int_{-\infty}^\infty \delta(y)^2\mathrm{d}y}{\exp(2T(\sigma^2/2+\gamma-\lambda))T^4\sigma^4}\right) = 0.$$

Remark. Obviously $a(U) := w_U(1)$ is the only possible definition. Thus, a describes the dependence of w_U on U at one.

Proof. We notice that (6.10) applies to the complex-valued functions and yields for $w_j := w_U$ and $w_k := \tilde{w}_U$ with the definitions (6.8) and (6.9) of f_U and g_U, respectively, and with $\theta := T\sigma^2 + T\gamma$

$$T^2e^{2T(\sigma^2/2+\gamma-\lambda)}\mathbb{E}\left[\int_0^1 w_U(u)\mathcal{L}_{\epsilon,U}(u)\mathrm{d}u\overline{\int_0^1 \tilde{w}_U(v)\mathcal{L}_{\epsilon,U}(v)\mathrm{d}v}\right]$$
$$= \epsilon^2U^4\int_{-\infty}^\infty \mathcal{F}(f_U(u)e^{T\sigma^2U^2u^2/2})(U(x-\theta))$$
$$\overline{\mathcal{F}(g_U(v)e^{T\sigma^2U^2v^2/2})(U(x-\theta))}\delta(x)^2\mathrm{d}x,$$
$$= 2\pi\epsilon^2U^3\int_{-\infty}^\infty f_U(u)e^{T\sigma^2U^2u^2/2}$$
$$\left(\overline{g_U(v)}e^{T\sigma^2U^2v^2/2} * \overline{\mathcal{F}^{-1}(\delta(y/U+\theta)^2)(v)}\right)(u)\mathrm{d}u$$
$$= 2\pi\epsilon^2U^4\int_0^1\int_0^1 f_U(u)\overline{g_U(v)}$$
$$\overline{\mathcal{F}^{-1}(\delta(y+\theta)^2)(U(u-v))}e^{T\sigma^2U^2(u^2+v^2)/2}\mathrm{d}u\mathrm{d}v. \tag{6.17}$$

For all $\delta > 0$ we have

$$\lim_{U\to\infty}T\sigma^2U^2e^{-T\sigma^2U^2/2}\int_{1-\delta/U}^1 ue^{T\sigma^2U^2u^2/2}\mathrm{d}u$$
$$= \lim_{U\to\infty}e^{-T\sigma^2U^2/2}\left[e^{T\sigma^2U^2u^2/2}\right]_{1-\delta/U}^1$$

$$
\begin{aligned}
&= 1 - \lim_{U\to\infty} e^{-T\sigma^2 U^2(1-(1-\delta/U)^2)/2} \\
&= 1 - \lim_{U\to\infty} e^{-T\sigma^2 U^2(2\delta/U-\delta^2/U^2)/2} \\
&= 1 - \lim_{U\to\infty} e^{-T\sigma^2 U\delta + T\sigma^2\delta^2/2} = 1. \qquad (6.18)
\end{aligned}
$$

For the product of two such sequences we obtain for all $\delta > 0$

$$
\lim_{U\to\infty} T^2\sigma^4 U^4 e^{-T\sigma^2U^2} \int_{1-\delta/U}^{1}\int_{1-\delta/U}^{1} uve^{T\sigma^2U^2(u^2+v^2)/2}\mathrm{d}u\mathrm{d}v = 1. \qquad (6.19)
$$

Likewise

$$
\lim_{U\to\infty} T^2\sigma^4 U^4 e^{-T\sigma^2U^2} \int_{0}^{1}\int_{0}^{1} uve^{T\sigma^2U^2(u^2+v^2)/2}\mathrm{d}u\mathrm{d}v = 1 \qquad (6.20)
$$

holds. We scale the integral in (6.17) appropriately:

$$
\begin{aligned}
&\lim_{U\to\infty}\Bigg(T^2\sigma^4U^4e^{-T\sigma^2U^2}\int_0^1\int_0^1 f_U(u)\overline{g_U(v)} \\
&\overline{\mathcal{F}^{-1}(\delta(y+\theta)^2)(U(u-v))}e^{T\sigma^2U^2(u^2+v^2)/2}\mathrm{d}u\mathrm{d}v \\
&- a(U)\overline{\tilde{a}(U)}\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(y)^2\mathrm{d}y\Bigg) \qquad (6.21) \\
= &\lim_{U\to\infty}\Bigg(T^2\sigma^4U^4e^{-T\sigma^2U^2}\int_0^1\int_0^1 uve^{T\sigma^2U^2(u^2+v^2)/2} \\
&\overline{\mathcal{F}^{-1}(\delta(y+\theta)^2)(U(u-v))}f_U(u)\overline{g_U(v)}/(uv)\mathrm{d}u\mathrm{d}v \\
&- a(U)\overline{\tilde{a}(U)}\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(y)^2\mathrm{d}y\Bigg) \qquad (6.22)
\end{aligned}
$$

We recall that in the Gaussian white noise model we assumed δ to be in $L^2(\mathbb{R})$. Since $\overline{\mathcal{F}^{-1}(\delta(y+\theta)^2)(U(u-v))}f_U(u)\overline{g_U(v)}/(uv)$ is bounded in $L^\infty([0,1]^2)$ independently of U for $U \geq 1$ and since the difference between (6.20) and (6.19) is zero, only the integral over $[1-\delta/U, 1]^2$ contributes to the limit. For all $\delta > 0$ the limit equals

$$
\begin{aligned}
= &\lim_{U\to\infty}\Bigg(T^2\sigma^4U^4e^{-T\sigma^2U^2}\int_{1-\delta/U}^1\int_{1-\delta/U}^1 uve^{T\sigma^2U^2(u^2+v^2)/2} \\
&\overline{\mathcal{F}^{-1}(\delta(y+\theta)^2)(U(u-v))}f_U(u)\overline{g_U(v)}/(uv)\mathrm{d}v\mathrm{d}u \\
&- a(U)\overline{\tilde{a}(U)}\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(y)^2\mathrm{d}y\Bigg) = 0, \qquad (6.23)
\end{aligned}
$$

which can be seen the following way. $\mathcal{F}^{-1}(\delta(y+\theta)^2)$ is continuous and $|U(u-v)| \leq \delta$ for all $u, v \in [1-\delta/U, 1]$. So by choosing δ small enough $\mathcal{F}^{-1}(\delta(y+\theta)^2)(U(u-v))$ gets arbitrarily close to $\mathcal{F}^{-1}(\delta(y+\theta)^2)(0) =$

$(1/2\pi)\int_{-\infty}^{\infty}\delta(y)^2\mathrm{d}y$. By (6.16), $w_U(u)$ tends to $a(U)$ and $\tilde{w}_U(v)$ tends to $\tilde{a}(U)$ for δ tending to zero. By choosing U large the factor $(-u+i/U)/\exp(T\mathcal{F}\mu(Uu))$ gets close to minus one for all $u\in[1-\delta/U,1]$. Thus, for small δ and large U the term $f_U(u)\overline{g_U(v)}/(uv)$ is close to $a(U)\overline{\tilde{a}(U)}$ for all $u,v\in[1-\delta/U,1]$.

Rescaling (6.17) and taking the limit $U\to\infty$ leads to

$$\begin{aligned}
&\lim_{U\to\infty}\Bigg(\frac{1}{\epsilon^2\exp(T\sigma^2U^2)}\mathbb{E}\left[\int_0^1 w_U(u)\mathcal{L}_{\epsilon,U}(u)\mathrm{d}u\overline{\int_0^1\tilde{w}_U(v)\mathcal{L}_{\epsilon,U}(v)\mathrm{d}v}\right]\\
&\quad-\frac{a(U)\overline{\tilde{a}(U)}\int_{-\infty}^{\infty}\delta(y)^2\mathrm{d}y}{\exp(2T(\sigma^2/2+\gamma-\lambda))T^4\sigma^4}\Bigg)\\
&=\lim_{U\to\infty}\Bigg(\frac{2\pi U^4\exp(-T\sigma^2U^2)}{T^2\exp(2T(\sigma^2/2+\gamma-\lambda))}\int_0^1\int_0^1 f_U(u)\overline{g_U(v)}\\
&\quad\overline{\mathcal{F}^{-1}(\delta(y+\theta_0)^2)(U(u-v))}e^{T\sigma^2U^2(u^2+v^2)/2}\mathrm{d}u\mathrm{d}v\\
&\quad-\frac{2\pi}{\exp(2T(\sigma^2/2+\gamma-\lambda))}\frac{a(U)\overline{\tilde{a}(U)}}{T^4\sigma^4 2\pi}\int_{-\infty}^{\infty}\delta(y)^2\mathrm{d}y\Bigg)=0,
\end{aligned}\tag{6.24}$$

where we used that (6.21) is zero. By (6.11) we have

$$\begin{aligned}
&T^2e^{2T(\sigma^2/2+\gamma-\lambda)}\mathbb{E}\left[\int_0^1 w_U(u)\mathcal{L}_{\epsilon,U}(u)\mathrm{d}u\int_0^1\tilde{w}_U(v)\mathcal{L}_{\epsilon,U}(v)\mathrm{d}v\right]\\
&=\epsilon^2U^4\int_{-\infty}^{\infty}\mathcal{F}(f_U(u)e^{T\sigma^2U^2u^2/2})(U(x-\theta))\\
&\quad\overline{\mathcal{F}(\overline{g_U(v)}e^{T\sigma^2U^2v^2/2})(-U(x-\theta))}\delta(x)^2\mathrm{d}x\\
&=\epsilon^2U^4\int_{-\infty}^{\infty}\mathcal{F}(f_U(u)e^{T\sigma^2U^2u^2/2})(U(x-\theta))\\
&\quad\overline{\mathcal{F}(\overline{g_U(-v)}e^{T\sigma^2U^2v^2/2})(U(x-\theta))}\delta(x)^2\mathrm{d}x\\
&=2\pi\epsilon^2U^3\int_{-\infty}^{\infty}f_U(u)e^{T\sigma^2U^2u^2/2}\\
&\quad\left(g_U(-v)e^{T\sigma^2U^2v^2/2}*\overline{\mathcal{F}^{-1}(\delta(y/U+\theta)^2)(v)}\right)(u)\mathrm{d}u\\
&=2\pi\epsilon^2U^4\int_0^1\int_{-1}^0 f_U(u)g_U(-v)\\
&\quad\overline{\mathcal{F}^{-1}(\delta(y+\theta)^2)(U(u-v))}e^{T\sigma^2U^2(u^2+v^2)/2}\mathrm{d}v\mathrm{d}u\\
&=2\pi\epsilon^2U^4\int_0^1\int_0^1 uve^{T\sigma^2U^2(u^2+v^2)/2}\\
&\quad\mathcal{F}^{-1}(\delta(y+\theta)^2)(-U(u+v))f_U(u)g_U(v)/(uv)\mathrm{d}v\mathrm{d}u.
\end{aligned}$$

Rescaling as in (6.24) leads to

$$\lim_{U\to\infty} \frac{1}{\epsilon^2 \exp(T\sigma^2 U^2)} \mathbb{E}\left[\int_0^1 w_U(u)\mathcal{L}_{\epsilon,U}(u)\mathrm{d}u \int_0^1 \tilde{w}_U(v)\mathcal{L}_{\epsilon,U}(v)\mathrm{d}v\right]$$
$$= \lim_{U\to\infty} \frac{2\pi U^4 \exp(-T\sigma^2 U^2)}{T^2 \exp(2T(\sigma^2/2+\gamma-\lambda))} \int_0^1 \int_0^1 uv e^{T\sigma^2 U^2 (u^2+v^2)/2}$$
$$\mathcal{F}^{-1}(\delta(y+\theta)^2)(-U(u+v)) f_U(u) g_U(v)/(uv)\mathrm{d}u\mathrm{d}v = 0, \tag{6.25}$$

since $\mathcal{F}^{-1}(\delta(y+\theta)^2)(u) \to 0$ for $|u| \to \infty$. $\square$

Lemma 3. *Let $\sigma > 0$ and $\delta \in L^\infty(\mathbb{R})$. Let $x_0 \in \mathbb{R}$ and for $j = 1, \dots, n$ let $w_j : [0,1] \to \mathbb{R}$ be continuous at one, Riemann-integrable and in $L^\infty([0,1])$. Then*

$$\frac{1}{\epsilon \exp(T\sigma^2 U^2/2)} \int_0^1 w_j(u)\mathcal{L}_{\epsilon,U}(u) e^{-iUux_0}\mathrm{d}u$$

converge jointly in distribution to

$$\frac{\|\delta\|_{L^2(\mathbb{R})} w_j(1)}{\sqrt{2}\exp(T(\sigma^2/2+\gamma-\lambda))T^2\sigma^2} (W + iV)$$

as $U \to \infty$, where W and V are independent standard normal random variables.

Proof. We define $w_U(u) := w_j(u)/\exp(iUux_0)$, $\tilde{w}_U(u) := w_k(u)/\exp(iUux_0)$, $a(U) := w_j(1)/\exp(iUx_0)$ and $\tilde{a}(U) := w_k(1)/\exp(iUx_0)$ and apply Lemma 2. Condition (6.16) is satisfied since w_j and w_k are continuous at one and since $\exp(-iUux_0) = \exp(-iUx_0)\exp(iU(1-u)x_0)$ where $U(1-u) \le \delta$ for $u \in [1-\delta/U, 1]$. We note that $a(U)\overline{\tilde{a}(U)} = w_j(1)w_k(1)$ is real. By Lemma 2 the covariances converge to the covariances of the claimed limit. The convergence in distribution follows by Lévy's continuity theorem. $\square$

Lemma 4. *Let $\sigma > 0$ and $\delta \in L^\infty(\mathbb{R})$. Let $w_1, w_2 : [0,1] \to \mathbb{R}$ be Riemann-integrable, in $L^\infty([0,1])$ and continuous at one. Let $x_1, x_2 \in \mathbb{R}$ and denote $x_2 - x_1$ by φ. Then*

$$\frac{1}{\epsilon e^{T\sigma^2 U^2/2}} \left(w_1(1) \int_0^1 \frac{w_2(u)\mathcal{L}_{\epsilon,U}(u)}{e^{iUux_2}}\mathrm{d}u - \frac{w_2(1)}{e^{iU\varphi}} \int_0^1 \frac{w_1(u)\mathcal{L}_{\epsilon,U}(u)}{e^{iUux_1}}\mathrm{d}u \right) \xrightarrow{\mathbb{Q}} 0,$$

as $U \to \infty$.

Proof. We define

$$w_U(u) := \frac{w_1(1)w_2(u)}{\exp(iUux_2)} - \frac{w_2(1)w_1(u)}{\exp(iU\varphi)\exp(iUux_1)}.$$

$w_U(u)$ fulfills condition (6.16) with $a(U) = 0$ for all $U \ge 1$. Lemma 2 yields

$$\lim_{U\to\infty} \frac{1}{\epsilon^2 \exp(T\sigma^2 U^2)} \mathbb{E}\left[\left|\int_0^1 w_U(u)\mathcal{L}_{\epsilon,U}(u)\mathrm{d}u\right|^2\right] = 0$$

and the statement follows by Lévy's continuity theorem. $\square$

6.3 The remainder term

In this section, we show that the contribution of the remainder term to the estimation vanishes asymptotically. We recall that the remainder term $\mathcal{R}_{\epsilon,U}$ depends on the Lévy triplet.

Lemma 5. *Let $\sigma_0 > 0$. Let $w_U \in L^\infty([0,1],\mathbb{C})$ be Riemann-integrable and let there be a constant $C > 0$ such that $\|w_U\|_\infty \leq C$ for all $U \geq 1$. If $\epsilon U(\epsilon)^2\sqrt{\log(U(\epsilon))}\exp(T\sigma_0^2 U(\epsilon)^2/2) \to 0$ as $\epsilon \to 0$, then for all Lévy triplets with $\sigma \leq \sigma_0$*

$$\frac{1}{\epsilon\exp(T\sigma_0^2U(\epsilon)^2/2)}\int_0^1 w_{U(\epsilon)}(u)\mathcal{R}_{\epsilon,U(\epsilon)}(u)\mathrm{d}u \xrightarrow{\mathbb{Q}} 0, \qquad \textit{as } \epsilon \to 0.$$

Proof. By the identity $\mathcal{R}_{\epsilon,U}(u) = (1/T)\log_{\geq\kappa^U(u)}(1+T\mathcal{L}_{\epsilon,U}(u)) - \mathcal{L}_{\epsilon,U}(u)$, where $\kappa^U(u) \leq 1/2$, we have to show that for $U = U(\epsilon)$

$$\frac{1}{\epsilon\exp(T\sigma_0^2U^2/2)}\int_0^1 w_U(u)\left(\log_{\geq\kappa^U(u)}(1+T\mathcal{L}_{\epsilon,U}(u)) - T\mathcal{L}_{\epsilon,U}(u)\right)\mathrm{d}u \tag{6.26}$$

converges in probability to zero.

For $z \in \mathbb{C}$ holds $\log(1+z) - z = O(|z|^2)$ as $|z| \to 0$. We define g by $g(z) := (\log(1+z)-z)/|z|^2$ for $z \neq 0$ and $g(0) := 0$. There are M and $\eta > 0$ such that $|g(z)| \leq M$ for all $|z| \leq \eta$. We may assume that $\eta \leq 1/2$. If the logarithm in the definition of g is replaced by the trimmed logarithm $\log_{\geq\kappa}$ with $\kappa \in (0,1/2]$ then g remains unchanged for $|z| \leq 1/2$. Thus, the statement holds uniformly for all $g_\kappa(z) := (\log_{\geq\kappa}(1+z)-z)/|z|^2$ with $\kappa \in]0,1/2]$.

By Proposition 1 we have $\sup_{u\in[-1,1]}|\mathcal{L}_{\epsilon,U}(u)| \xrightarrow{\mathbb{Q}} 0$. Let $\tau > 0$ be given. Eventually we have

$$\begin{aligned}&\mathbb{Q}\left(\exists u \in [-1,1] : |\log_{\geq\kappa^U(u)}(1+T\mathcal{L}_{\epsilon,U}(u)) - T\mathcal{L}_{\epsilon,U}(u)| > MT^2|\mathcal{L}_{\epsilon,U}(u)|^2\right)\\ &\leq \mathbb{Q}\left(T\sup_{u\in[-1,1]}|\mathcal{L}_{\epsilon,U}(u)| > \eta\right) < \tau.\end{aligned}$$

Except on a set with probability less than τ we have eventually

$$\begin{aligned}&\frac{1}{\epsilon\exp(T\sigma_0^2U^2/2)}\left|\int_0^1 w_U(u)\left(\log_{\geq\kappa^U(u)}(1+T\mathcal{L}_{\epsilon,U}(u)) - T\mathcal{L}_{\epsilon,U}(u)\right)\mathrm{d}u\right|\\ &\leq \frac{MT^2}{\epsilon\exp(T\sigma_0^2U^2/2)}\int_0^1|w_U(u)\mathcal{L}_{\epsilon,U}(u)^2|\mathrm{d}u. \end{aligned}\tag{6.27}$$

Hence (6.26) converges in probability to zero if (6.27) converges in probability to zero. The convergence

$$\frac{1}{\epsilon\exp(T\sigma_0^2U^2/2)}\int_0^1|w_U(u)\mathcal{L}_{\epsilon,U}(u)^2|\mathrm{d}u \to 0$$

holds even in L^1 since

$$
\begin{aligned}
&\frac{1}{\epsilon \exp(T\sigma_0^2 U^2/2)} \mathbb{E}\left[\int_0^1 |w_U(u)\mathcal{L}_{\epsilon,U}(u)^2| \mathrm{d}u\right] && (6.28)\\
&\leq \frac{C}{\epsilon \exp(T\sigma_0^2 U^2/2)} \mathbb{E}\left[\int_0^1 |\mathcal{L}_{\epsilon,U}(u)^2| \mathrm{d}u\right]\\
&\leq \frac{C}{\epsilon \exp(T\sigma_0^2 U^2/2)}\\
&\quad \int_0^1 \left|\frac{\epsilon\, iUu(1+iUu)}{T(1+iUu(1+iUu)\mathcal{F}\mathcal{O}(Uu))}\right|^2 \mathbb{E}\left[\left|\int_{-\infty}^{\infty} e^{iUux}\delta(x)\mathrm{d}W(x)\right|^2\right] \mathrm{d}u\\
&\leq \frac{C}{\epsilon \exp(T\sigma_0^2 U^2/2)} \int_0^1 \frac{\epsilon^2 (U^2+U^4) u \exp(T\sigma^2 U^2 u^2) \|\delta\|_{L^2(\mathbb{R})}^2}{T^2 \exp(2T(\sigma^2/2+\gamma-\lambda) - 2T\|\mathcal{F}\mu\|_\infty)} \mathrm{d}u, && (6.29)
\end{aligned}
$$

for $\sigma = 0$ this converges to zero and for $\sigma > 0$ we further calculate,

$$
\begin{aligned}
&= \frac{C\epsilon(1+U^2)\|\delta\|_{L^2(\mathbb{R})}^2 \int_0^1 2T\sigma^2 U^2 u \exp(T\sigma^2 U^2 u^2)\mathrm{d}u}{\exp(T\sigma_0^2 U^2/2) 2T^3\sigma^2 \exp(2T(\sigma^2/2+\gamma-\lambda) - 2T\|\mathcal{F}\mu\|_\infty)}\\
&= \frac{C\epsilon(1+U^2)\|\delta\|_{L^2(\mathbb{R})}^2 (\exp(T\sigma^2 U^2) - 1)}{\exp(T\sigma_0^2 U^2/2) 2T^3\sigma^2 \exp(2T(\sigma^2/2+\gamma-\lambda) - 2T\|\mathcal{F}\mu\|_\infty)}\\
&\leq \frac{C\|\delta\|_{L^2(\mathbb{R})}^2 \epsilon(1+U^2)(\exp(T\sigma_0^2 U^2/2))}{2T^3\sigma^2 \exp(2T(\sigma^2/2+\gamma-\lambda) - 2T\|\mathcal{F}\mu\|_\infty)} \to 0
\end{aligned}
$$

as $\epsilon \to 0$. Thus, (6.26) converges in probability to zero. $\square$

Lemma 6. *Let $w_U \in L^\infty([0,1],\mathbb{C})$ be Riemann-integrable and let there be a constant $C > 0$ such that $\|w_U\|_\infty \leq C$ for all $U \geq 1$. If $U(\epsilon) \to \infty$ and $\epsilon U(\epsilon)^{5/2} \to 0$ as $\epsilon \to 0$, then for all Lévy triplets with $\sigma = 0$*

$$
\frac{1}{\epsilon U(\epsilon)^{3/2}} \int_0^1 w_{U(\epsilon)}(u)\mathcal{R}_{\epsilon,U(\epsilon)}(u)\mathrm{d}u \xrightarrow{\mathbb{Q}} 0,
$$

as $\epsilon \to 0$.

Proof. We follow the proof of Lemma 5. $\sup_{u\in[-1,1]} |\mathcal{L}_{\epsilon,U}(u)| \xrightarrow{\mathbb{Q}} 0$ holds by Proposition 1. We set $\sigma_0 = 0$ and divide by $U^{3/2}$ in (6.26) and (6.27). Then we use that (6.28) is bounded by (6.29), where we set $\sigma_0 = \sigma = 0$ and divide by $U^{3/2}$ again. We obtain

$$
\frac{1}{\epsilon U^{3/2}} \mathbb{E}\left[\int_0^1 |w_U(u)\mathcal{L}_{\epsilon,U}(u)^2| \mathrm{d}u\right] \leq \frac{C\epsilon\,(U^{1/2}+U^{5/2})\|\delta\|_{L^2(\mathbb{R})}^2}{T^2 \exp(T(2(\gamma-\lambda) - 2\|\mathcal{F}\mu\|_\infty))} \to 0
$$

as $\epsilon \to 0$, which implies the desired convergence. $\square$

6.4 The approximation errors

The approximation error can be controlled as in [2] using the order conditions (2.9) on the weight functions. The Lévy triplet $\mathcal{T} = (\sigma^2, \gamma, \mu)$ was assumed to be contained in $\mathcal{G}_s(R, \sigma_{\max})$, especially μ is s-times weakly differentiable and $\max_{0\le k\le s} \|\mu^{(k)}\|_{L^2(\mathbb{R})} \le R$, $\|\mu^{(s)}\|_\infty \le R$.

We use $(iu)^s \mathcal{F}\mu(u) = \mathcal{F}\mu^{(s)}(u)$ and the Plancherel identity to bound the approximation error by

$$\begin{aligned}
&\left|\frac{2}{U^2}\int_0^1 \mathrm{Re}(\mathcal{F}\mu(Uu)) w_\sigma^1(u)\mathrm{d}u\right| \\
&= \frac{1}{U^2}\left|\int_{-1}^1 \mathcal{F}\mu(Uu) w_\sigma^1(u)\mathrm{d}u\right| \\
&= \frac{2\pi}{U^2}\left|\int_{-\infty}^{\infty} \mu^{(s)}(x/U) U^{-1}\overline{\mathcal{F}^{-1}(w_\sigma^1(u)/(iUu)^s)(x)}\mathrm{d}x\right| \\
&\le U^{-(s+3)}\|\mu^{(s)}\|_\infty \|\mathcal{F}(w_\sigma^1(u)/u^s)\|_{L^1(\mathbb{R})}.
\end{aligned} \tag{6.30}$$

Analogously we obtain

$$\left|\frac{2}{U}\int_0^1 \mathrm{Im}(\mathcal{F}\mu(Uu)) w_\gamma^1(u)\mathrm{d}u\right| \le U^{-(s+2)}\|\mu^{(s)}\|_\infty \|\mathcal{F}(w_\gamma^1(u)/u^s)\|_{L^1(\mathbb{R})}, \tag{6.31}$$

$$\left|2\int_0^1 \mathrm{Re}(\mathcal{F}\mu(Uu)) w_\lambda^1(u)\mathrm{d}u\right| \le U^{-(s+1)}\|\mu^{(s)}\|_\infty \|\mathcal{F}(w_\lambda^1(u)/u^s)\|_{L^1(\mathbb{R})}. \tag{6.32}$$

The last error term in (6.3) can be bounded by

$$\begin{aligned}
&\left|U\mathcal{F}^{-1}\left[(1-w_\mu^1(u))\mathcal{F}\mu(Uu)\right](Ux)\right| \\
&= \frac{U}{2\pi}\left|\int_{-\infty}^{\infty}(1-w_\mu^1(u))\mathcal{F}\mu(Uu)e^{-iUux}\mathrm{d}u\right| \\
&= \frac{1}{2\pi U^{s-1}}\left|\int_{-\infty}^{\infty}\overline{\frac{1-w_\mu^1(u)}{u^s}e^{iUux}}\mathcal{F}\mu^{(s)}(Uu)\mathrm{d}u\right| \\
&= U^{-s}\left|\int_{-\infty}^{\infty}\overline{\mathcal{F}^{-1}\left(\frac{1-w_\mu^1(u)}{u^s}e^{iUux}\right)(y)}\mu^{(s)}\left(\frac{y}{U}\right)\mathrm{d}y\right| \\
&\le \frac{\|\mu^{(s)}\|_\infty}{2\pi U^s}\left\|\mathcal{F}\left(\frac{1-w_\mu^1(u)}{u^s}\right)\right\|_{L^1(\mathbb{R})}.
\end{aligned} \tag{6.33}$$

6.5 Proof of Theorem 1

We consider the decompositions of the estimation errors (6.1)–(6.3). The undersmoothing $\epsilon U(\epsilon)^{(2s+5)/2} \to \infty$ is equivalent to $U(\epsilon)^{-(s+3)} = o(\epsilon U(\epsilon)^{-1/2})$

and implies in view of (6.30) that the approximation error of $\hat{\sigma}^2$ is asymptotically negligible. Likewise the error terms $2U^{-1}\int_0^1 \mathrm{Im}(\mathcal{F}\mu(Uu))w_\gamma^1(u)\mathrm{d}u$, $2\int_0^1 \mathrm{Re}(\mathcal{F}\mu(Uu))w_\lambda^1(u)\mathrm{d}u$ and $U\mathcal{F}^{-1}\left[(1-w_\mu^1(u))\mathcal{F}\mu(Uu)\right](Ux)$ can be bounded as in (6.31), (6.32) and (6.33) and vanish asymptotically. Since $\hat{\sigma}^2$ converges with a faster rate than $\hat{\gamma}$ and $\hat{\gamma}$ converges with a faster rate than $\hat{\lambda}$, the error $\Delta\hat{\sigma}^2$ vanishes asymptotically in (6.1) and in (6.2) as well as $\Delta\hat{\gamma}$ is asymptotically negligible in (6.2). For $x \neq 0$ we can apply the Riemann-Lebesgue lemma to the second, the third and the fourth error term in (6.3) and we see that they are of order $o_{\mathbb{Q}}(\epsilon U(\epsilon)^{5/2})$. For $x = 0$ due to the symmetry of w_μ^1 the third term vanishes asymptotically but the second and the fourth term do not. The error terms of $\hat{\mu}(x)$ we have to consider are in the case $x \neq 0$

$$
\begin{aligned}
&U\mathcal{F}^{-1}\left[\Delta\psi_\epsilon(Uu)w_\mu^1(u)\right](Ux)\\
&= \frac{U}{2\pi}2\int_0^1 w_\mu^1(u)\mathrm{Re}\left(\Delta\psi_\epsilon(Uu)e^{-iUux}\right)\mathrm{d}u
\end{aligned}
$$

and in the case $x = 0$

$$
\begin{aligned}
&\mathcal{F}^{-1}\left[\Delta\psi_\epsilon(Uu))w_\mu^1(u)\right](0)\\
&+\int_0^1 \mathrm{Re}(\Delta\psi_\epsilon(Uu))w_\sigma^1(u)\mathrm{d}u\,\mathcal{F}^{-1}\left[u^2 w_\sigma^1(u)\right](0)\\
&-2\int_0^1 \mathrm{Re}(\Delta\psi_\epsilon(Uu))w_\lambda^1(u)\mathrm{d}u\,\mathcal{F}^{-1}\left[w_\mu^1(u)\right](0)\\
=&\frac{1}{2\pi}2\int_0^1 \mathrm{Re}(\Delta\psi_\epsilon(Uu))w_0(u)\mathrm{d}u.
\end{aligned}
$$

(2.9) implies that w_σ^1, w_γ^1, w_λ^1 and w_μ^1 are continuous and bounded, especially they are Riemann-integrable and in $L^\infty([-1,1])$. Applying Lemma 1 to the linearized stochastic errors and Lemma 6 to the remainder terms yields the theorem.

6.6 Proof of Theorem 2

To see the first line we set $x_1 = x_2 = 0$, $w_1 \equiv 1$ and $w_2 = w_\sigma$ in Lemma 4 and $w_U = w_\sigma^1$ in Lemma 5. The second and third line follow analogously. In order to derive the last line we observe

$$
\begin{aligned}
&\mathcal{F}^{-1}\left[\Delta\psi_\epsilon(Uu)w_\mu^1(u)\right](Ux)\\
&= 2\int_0^1 \mathrm{Re}(\Delta\psi_\epsilon(Uu)e^{-iUux})w_\mu^1(u)\mathrm{d}u/(2\pi)
\end{aligned}
$$

and apply Lemma 4 with $x_1 = 0$, $x_2 = x$, $w_1 \equiv 1$ and $w_2 = w_\mu^1$, . The remainder term vanishes by setting $w_U(u) = w_\mu^1(u)e^{-iUux}$ in Lemma 5.

6.7 Proof of Theorem 3

By the undersmoothing $\epsilon U^{s+1}\exp(T\sigma^2U^2/2)\to\infty$ we have $U^{-(s+3)}=o(\epsilon U^{-2}\exp(T\sigma^2U^2/2))$ so that the approximation error of $\hat{\sigma}$ vanishes. A similar reasoning applies to the approximation errors of the other estimators. Since $\hat{\sigma}$ converges with a faster rate than $\hat{\gamma}$ and $\hat{\gamma}$ with a faster rate than $\hat{\lambda}$ the leading stochastic error terms are given in Theorem 2 and the convergence of the first three lines follows by this theorem. For $x\neq 0$ all stochastic errors in (6.3) are negligible except the first one. We obtain the convergence in the last line by Theorem 2. We observe that $\mathcal{F}^{-1}[uw_\mu^1(u)](0)=0$, since w_μ^1 is symmetric. For $x=0$ the relevant stochastic error terms are

$$
\begin{aligned}
&\mathcal{F}^{-1}\left[\Delta\psi_\epsilon(Uu)w_\mu^1(u)\right](0)\\
&+\int_0^1 \mathrm{Re}(\Delta\psi_\epsilon(Uu))w_\sigma^1(u)\mathrm{d}u\,\mathcal{F}^{-1}\left[u^2w_\mu^1(u)\right](0)\\
&-2\int_0^1 \mathrm{Re}(\Delta\psi_\epsilon(Uu))w_\lambda^1(u)\mathrm{d}u\,\mathcal{F}^{-1}\left[w_\mu^1(u)\right](0)\\
=&\frac{1}{2\pi}2\int_0^1 \mathrm{Re}(\Delta\psi_\epsilon(Uu))w_0(u)\mathrm{d}u.
\end{aligned}
$$

We apply Lemma 4 with $x_1=x_2=0$, $w_1\equiv 1$ and $w_2=w_0$ to this term. The remainder term converges to zero by Lemma 5. This shows the convergence in the next to last line.

7 Tests and a confidence set for the volatility

In this section, we test the hypotheses $H_0:\sigma=\sigma_0$ against the alternative $H_1:|\sigma-\sigma_0|\geq\tau$, $\tau>0$, and construct a confidence set for σ. We assume that $\mathcal{T}\in\mathcal{H}_s(R,\sigma_{\max})$ especially we assume that $\sigma\in[0,\sigma_{\max}]$.

For $\sigma_0>0$ the most natural test statistic is the following. In order to apply the uniform version of Theorem 3 under H_0 we choose a cut-off value $U(\epsilon)$ such that $\epsilon U(\epsilon)^2\log(U(\epsilon))^{1/2}\exp(T\sigma_0^2U(\epsilon)^2/2)\to 0$ and $\epsilon U(\epsilon)^{s+1}\exp(T\sigma_0^2U(\epsilon)^2/2)\to\infty$ as $\epsilon\to 0$. Let $\hat{\sigma}^2$ be the estimator corresponding to this cut-off value $U(\epsilon)$. We ensure $\hat{\sigma}^2\in[0,\sigma_{\max}^2]$ by taking the maximum with zero and the minimum with $\sigma_{\max}^2$. Likewise we ensure $\hat{\gamma}\in[-R,R]$ and $\hat{\lambda}\in[0,R]$. We choose a weight function with $w_\sigma^1(1)\neq 0$ and define

$$
S_{\sigma_0}:=\frac{U(\epsilon)^2(\hat{\sigma}^2-\sigma_0^2)}{\hat{d}_{\sigma_0}\,\epsilon\exp(T\sigma_0^2U(\epsilon)^2/2)},\qquad \hat{d}_{\sigma_0}:=\frac{\sqrt{2}\,|w_\sigma^1(1)|\,\|\delta\|_{L^2(\mathbb{R})}}{\exp(T(\sigma_0^2/2+\hat{\gamma}-\hat{\lambda}))T^2\sigma_0^2}.
$$

Under H_0 the test statistic S_{σ_0} converges uniformly in distribution to a standard normal random variable by the uniform version of Theorem 3. We

decompose

$$S_{\sigma_0} = \frac{U(\epsilon)^2 \Delta \hat{\sigma}^2}{\hat{d}_{\sigma_0}\, \epsilon \exp(T\sigma_0^2 U(\epsilon)^2/2)} + \frac{U(\epsilon)^2(\sigma^2 - \sigma_0^2)}{\hat{d}_{\sigma_0}\, \epsilon \exp(T\sigma_0^2 U(\epsilon)^2/2)}. \tag{7.1}$$

We will show that for $\sigma \leq \sigma_0 - \tau$ the first term converges uniformly in probability to zero. The approximation error contributes a term that converges deterministically to zero by the bound (6.30) and by the assumption $\epsilon U(\epsilon)^{s+1} \exp(T\sigma_0^2 U(\epsilon)^2/2) \to \infty$ as $\epsilon \to 0$. The remainder term of the stochastic error in the first summand of (7.1) converge uniformly in probability to zero by Subsection 8.3 and the linearized stochastic error by the following lemma.

Lemma 7. *Let $\delta \in L^\infty(\mathbb{R})$. Let $w_U \in L^\infty([0,1], \mathbb{C})$ be Riemann-integrable and let there be a constant $C > 0$ such that $\|w_U\|_\infty \leq C$ for all $U \geq 1$. Then for all $\eta, \tau > 0$*

$$\sup_{\mathcal{T} \in \mathcal{H}_s(R, \sigma_0 - \tau)} \mathbb{Q}_{\mathcal{T}} \left(\left| \frac{1}{\epsilon \exp(T\sigma_0^2 U^2/2)} \int_0^1 w_U(u) \mathcal{L}_{\epsilon,U}(u) \mathrm{d}u \right| > \eta \right) \to 0,$$

as $U \to \infty$

Proof. It is equivalent to consider for all $\tau' > 0$ the supremum over $\mathcal{H}_s(R, \sqrt{\sigma_0^2 - \tau'})$, such that $\sigma^2 \leq \sigma_0^2 - \tau'$ is satisfied for each $\mathcal{T}$. By (6.17) we have

$$\begin{aligned}
&\sup_{\mathcal{T}} \frac{1}{\epsilon^2 \exp(T\sigma_0^2 U^2)} \mathbb{E}\left[\int_0^1 w_U(u) \mathcal{L}_{\epsilon,U}(u) \mathrm{d}u \overline{\int_0^1 w_U(v) \mathcal{L}_{\epsilon,U}(v) \mathrm{d}v} \right] \\
&= \sup_{\mathcal{T}} \frac{2\pi U^4 \exp(2T(\lambda - \gamma - \sigma^2/2))}{T^2 \exp(T\sigma_0^2 U^2)} \int_0^1 \int_0^1 f_U(u) \overline{f_U(v)} \\
&\quad \overline{\mathcal{F}^{-1}(\delta(y+\theta)^2)(U(u-v))} e^{T\sigma^2 U^2 (u^2+v^2)/2} \mathrm{d}u \mathrm{d}v \\
&\leq \frac{4\pi U^4 \exp(6TR) C^2}{T^2 \exp(T\tau' U^2)} \|\mathcal{F}^{-1}\delta^2\|_\infty \to 0,
\end{aligned}$$

as $U \to \infty$, where we used $\|f_U\|_\infty \leq \sqrt{2} C \exp(TR)$. $\square$

We have seen that for $\sigma \leq \sigma_0 - \tau$ the first term in (7.1) converges to zero uniformly in probability. The second term is $\hat{d}_{\sigma_0}^{-1}$ times a deterministic sequence converging to $-\infty$. We note that $\hat{d}_{\sigma_0}$ is bounded from above and below. Consequently it holds $\lim_{\epsilon \to 0} \inf_{\mathcal{H}_s(R, \sigma_0 - \tau)} \mathbb{Q}(S_{\sigma_0} < c) = 1$ for all $c \in \mathbb{R}$. We would like to make a similar statement in the case $\sigma \geq \sigma_0 + \tau$. Unfortunately for $\sigma > \sigma_0$ the variance of $\mathcal{L}_{\epsilon, U(\epsilon)}(1)$ does not converge to zero. So it is not possible to find a bound like in Proposition 1, which converges to zero. Consequently the remainder term cannot be controlled by Lemma 5. We modify the test statistic in the following way. We choose $\bar{\sigma} > \sigma_{\max}$ and

let $\bar{U}(\epsilon)$ be a cut-off value with $\epsilon\bar{U}(\epsilon)^2\log(\bar{U}(\epsilon))^{1/2}\exp(T\bar{\sigma}^2\bar{U}(\epsilon)^2/2)\to 0$ and $\epsilon\bar{U}(\epsilon)^{s+1}\exp(T\bar{\sigma}^2\bar{U}(\epsilon)^2/2)\to\infty$ as $\epsilon\to 0$. We further assume that

$$\frac{\bar{U}(\epsilon)^2\exp(-T\bar{\sigma}^2\bar{U}(\epsilon)^2/2)}{U(\epsilon)^2\exp(-T\sigma_0^2U(\epsilon)^2/2)}\to\infty. \tag{7.2}$$

This can for example be ensured by choosing the cut-off values $U(\epsilon)$ and $\bar{U}(\epsilon)$ according to (3.6) with α and $\bar{\alpha}>\alpha$, respectively. Let $\tilde{\sigma}^2$ be the estimator of σ^2 corresponding to the cut-off value $\bar{U}(\epsilon)$. We define

$$\tilde{S}_{\sigma_0} := S_{\sigma_0} + \frac{\bar{U}(\epsilon)^2(\tilde{\sigma}^2-\sigma_0^2)}{\epsilon\exp(T\bar{\sigma}^2\bar{U}(\epsilon)^2/2)} \tag{7.3}$$

$$= S_{\sigma_0} + \frac{\bar{U}(\epsilon)^2(\tilde{\sigma}^2-\sigma^2)}{\epsilon\exp(T\bar{\sigma}^2\bar{U}(\epsilon)^2/2)} + \frac{\bar{U}(\epsilon)^2(\sigma^2-\sigma_0^2)}{\epsilon\exp(T\bar{\sigma}^2\bar{U}(\epsilon)^2/2)}. \tag{7.4}$$

Under H_0 the statistic S_{σ_0} can be written as in Lemma 8. The second term in (7.3) converges uniformly in probability to zero. Thus, under H_0 the modified statistic $\tilde{S}_{\sigma_0}$ converges uniformly in distribution to a standard normal random variable by Lemma 8. For $\sigma\leq\sigma_0-\tau$ the second term in (7.4) converges uniformly in probability to zero and the third term is deterministic sequence converging to $-\infty$. As for S_{σ_0} it holds $\lim_{\epsilon\to 0}\inf_{\mathcal{H}_s(R,\sigma_0-\tau)}\mathbb{Q}(\tilde{S}_{\sigma_0}<c)=1$ for all $c\in\mathbb{R}$. But now we are also able to make a similar statement for $\sigma\geq\sigma_0+\tau$. Since we bounded the estimators S_{σ_0} cannot diverge faster than $\epsilon^{-1}U(\epsilon)^2\exp(-T\sigma_0^2U(\epsilon)^2/2)$. For $\sigma\in[0,\sigma_{\max}]$ the second term in (7.4) converges uniformly in probability to zero. Owing to (7.2) the third term in (7.4) tends to infinity faster than the bound of S_{σ_0}. It holds $\lim_{\epsilon\to 0}\inf_{\mathcal{T}}\mathbb{Q}(\tilde{S}_{\sigma_0}>c)=1$ for all $c\in\mathbb{R}$, where the infimum is over all $\mathcal{T}\in\mathcal{H}_s(R,\sigma_{\max})$ with $\sigma\geq\sigma_0+\tau$.

Let q_α denote the $(1-\alpha)$–quantile of the standard normal distribution. For $\sigma_0\in(0,\sigma_{\max})$ we define the tests

$$\varphi_{\sigma_0} := \begin{cases} 0, & \text{if } |\tilde{S}_{\sigma_0}|\leq q_{\alpha/2} \\ 1, & \text{if } |\tilde{S}_{\sigma_0}|> q_{\alpha/2}\end{cases}, \qquad \varphi_{\sigma_{\max}} := \begin{cases} 0, & \text{if } S_{\sigma_{\max}}\geq q_{1-\alpha} \\ 1, & \text{if } S_{\sigma_{\max}}< q_{1-\alpha}\end{cases}.$$

For $\sigma_0=0$ we would like to apply Theorem 1. To this end we choose the cut-off value $U(\epsilon)$ such that $\epsilon U(\epsilon)^{5/2}\to 0$ and $\epsilon U(\epsilon)^{(2s+5)/2}\to\infty$. Let $\hat{\sigma}^2$ be the estimator corresponding to this cut-off value $U(\epsilon)$, where $\hat{\sigma}^2\in[0,\sigma_{\max}^2]$ is ensured. We assume that δ is positive on $[-TR,TR]$ and define

$$S_0 := \frac{U(\epsilon)^{1/2}\hat{\sigma}^2}{\hat{d}_0\,\epsilon} + \frac{\bar{U}(\epsilon)^2\tilde{\sigma}^2}{\epsilon\exp(T\sigma_{\max}^2\bar{U}(\epsilon)^2/2)},$$

$$\hat{d}_0 := \frac{2\sqrt{\pi}\delta(T\hat{\gamma})}{\exp(T(\hat{\gamma}-\hat{\lambda}))T}\left(\int_0^1 u^4 w_\sigma^1(u)^2\mathrm{d}u\right)^{1/2}.$$

Under $H_0:\sigma^2=0$ the test statistic S_0 converges uniformly in distribution to a standard normal random variable by a similar argument as for $\tilde{S}_{\sigma_0}$.

We observe that the first term of S_0 is nonnegative. Under $H_1 : \sigma \geq \tau$ the second term of S_0 may be decomposed as in (7.4) into a part that converges uniformly in distribution to zero and into a deterministic sequence converging to infinity. It holds $\lim_{\epsilon\to 0} \inf_{\mathcal{T}} \mathbb{Q}(S_0 > c) = 1$ for all $c \in \mathbb{R}$, where the infimum is over all $\mathcal{T} \in \mathcal{H}_s(R, \sigma_{\max})$ with $\sigma \geq \tau$. We define the test

$$\varphi_0 := \begin{cases} 0, & \text{if } S_0 \leq q_\alpha \\ 1, & \text{if } S_0 > q_\alpha \end{cases}.$$

Since we have a test on $\sigma = \sigma_0$ for each $\sigma_0 \in [0, \sigma_{\max}]$ we may use this family of tests to define a confidence set for σ. We define $M_\epsilon := \{\sigma | \varphi_\sigma = 0\}$ and obtain $\lim_{\epsilon\to 0} \inf_{\mathcal{T}} \mathbb{Q}_{\mathcal{T}}(\sigma \in M_\epsilon) = 1 - \alpha$, where the infimum is over all $\mathcal{T} \in \mathcal{H}_s(R, \sigma_{\max})$ with volatility σ. The set M_ϵ is not necessarily an interval. For $\sigma_0 \in (0, \sigma_{\max})$ the cut-off value $U(\epsilon)$ may be chosen as a continuous function of σ_0 by (3.6). The estimators $\hat{\sigma}^2$, $\hat{\gamma}$ and $\hat{\lambda}$ depend continuously on the cut-off value $U(\epsilon)$, which can be seen by substituting $v = u/U$ in (2.3), (2.4) and (2.5) and applying the continuity theorem on parameter dependent integrals. Thus, $\tilde{S} : (0, \sigma_{\max}) \to \mathbb{R}$, $\sigma \mapsto \tilde{S}_\sigma$ is continuous and $M_\epsilon \cap (0, \sigma_{\max})$ may be written as the preimage $\tilde{S}^{-1}([-q_{\alpha/2}, q_{\alpha/2}])$ of the continuous function $\tilde{S}$.

8 Uniform convergence

The asymptotic normality results hold for each Lévy triplet $\mathcal{T} \in \mathcal{G}_s(R, \sigma_{\max})$. The speed of convergence might depend on $\mathcal{T}$. To make statements on confidence sets and on hypotheses tests it is useful to control the speed of convergence uniformly over a class of Lévy triplets. We fix some arbitrarily slowly decreasing function h with $h(u) \to 0$ as $|u| \to \infty$. We will show uniform convergence for the class $\mathcal{H}_s(R, \sigma_{\max})$ consisting of all Lévy triplets in $\mathcal{G}_s(R, \sigma_{\max})$ satisfying the additional conditions (4.2), which we recall to be

$$\|\mathcal{F}\mu\|_\infty \leq R, \quad |\mathcal{F}\mu(u)| \leq R\, h(u), \quad \forall u \in \mathbb{R}.$$

The first condition can easily be ensured by $\|\mu\|_{L^1(\mathbb{R})} \leq R$. For $h(u) = |u|^{-1}$ the second condition can be ensured by $\|\mu^{(1)}\|_{L^1(\mathbb{R})} \leq R$. For each Lévy triplet in $\mathcal{G}_s(R, \sigma_{\max})$ the function $\mathcal{F}\mu(u)$ tends to zero as $|u| \to 0$ by the Riemann-Lebesgue lemma. Especially for each $\mathcal{T} \in \mathcal{G}_s(R, \sigma_{\max})$ there are h and $R' > 0$ such that $\mathcal{T} \in \mathcal{H}_s(R', \sigma_{\max})$.

In the case $\sigma > 0$ some covariances do not converge. We show uniform convergence for the joint distribution only in such cases, where for $\sigma > 0$ the covariances do converge. As it turns out it is also important that the covariance matrix of the limit is nondegenerated. We cover uniform convergence of $\hat{\sigma}^2$, $\hat{\gamma}$, $\hat{\lambda}$, $\hat{\mu}(x)$ and of $(\hat{\gamma}, \hat{\lambda})$.

A sequence of random variables X_n converges to X *in total variation* if

$$\sup_B |\mathbb{P}(X_n \in B) - \mathbb{P}(X \in B)| \to 0$$

as $n \to \infty$, where the supremum is taken over all measurable sets B. For sequences $X_{\vartheta,n}$, $\vartheta \in \theta$, we say that they converge to X *in total variation uniformly over* θ if

$$\sup_{\vartheta\in\theta} \sup_B |\mathbb{P}(X_{\vartheta,n} \in B) - \mathbb{P}(X \in B)| \to 0$$

as $n \to \infty$. Motivated by the Portmanteau theorem we say for $X_{\vartheta,n}$, $\vartheta \in \theta$, and X with values in some metric space that $X_{\vartheta,n}$ converge to X *in distribution uniformly over* θ if for all Borel sets B with $\mathbb{P}(X \in \partial B) = 0$ we have

$$\sup_{\vartheta\in\theta} |\mathbb{P}(X_{\vartheta,n} \in B) - \mathbb{P}(X \in B)| \to 0 \qquad \text{as } n \to \infty.$$

Theorem 4. *Let the assumptions of Theorem 1 be fulfilled and let δ be continuous and positive. Each marginal convergence in Theorem 1 is a uniform convergence in distribution over $\mathcal{H}_s(R, 0)$ if the standard deviation is positive and both sides are divided by it.*

Let $\sigma > 0$ and let the assumptions of Theorem 3 be fulfilled. Each marginal convergences in Theorem 3 is a uniform convergence in distribution over all $\mathcal{T} \in \mathcal{H}_s(R, \sigma_{\max})$ with volatility σ if the standard deviation is positive and both sides are divided by it.

Remark. In both cases uniform convergence in distribution does also hold for $\hat{\gamma}$ and $\hat{\lambda}$ jointly and in the standard deviation on the left side γ and λ may be replaced by their estimators.

The following lemma may be seen as a generalization of Slutsky's lemma for uniform convergence. It is the key step to show the uniform convergence in distribution in Theorem 4.

Lemma 8. *Let $X_{\vartheta,n}$, $Y_{\vartheta,n}$, $\vartheta \in \theta$, $n \in \mathbb{N}$, and X be random vectors such that $X_{\vartheta,n}$ converge to X in total variation uniformly over θ and $\sup_{\vartheta\in\theta} \mathbb{P}(|Y_{\vartheta,n}| \geq \delta) \to 0$ as $n \to \infty$ for all $\delta > 0$. Let $Z_{\vartheta,n}$ be random variables with $\sup_{\vartheta\in\theta} \mathbb{P}(|Z_{\vartheta,n} - 1| \geq \delta) \to 0$ for all $\delta > 0$. Then $Z_{\vartheta,n} X_{\vartheta,n} + Y_{\vartheta,n}$ converge to X in distribution uniformly over θ.*

Proof. For $\delta > 0$ we define

$$B^{\delta} := \{y \in \mathbb{R}^d |\ |x - y| < \delta \text{ for some } x \in B\},$$
$$B^{-\delta} := \{y \in \mathbb{R}^d | x \in B \text{ for all } x \text{ with } |x - y| < \delta\}.$$

As B^{δ} and $B^{-\delta}$ are open and closed, respectively, they are Borel sets. It holds $\bigcap_{\delta>0} B^{\delta} = \overline{B}$ and $\bigcup_{\delta>0} B^{-\delta} = B^{\circ}$. For B with $\mathbb{P}(X \in \partial B) = 0$, we have $\mathbb{P}(X \in B^{\circ}) = \mathbb{P}(X \in \overline{B}) = \mathbb{P}(X \in B)$. Consequently

$$\lim_{\delta\to 0} \mathbb{P}(X \in B^{\delta}) = \lim_{\delta\to 0} \mathbb{P}(X \in B^{-\delta}) = \mathbb{P}(X \in B). \tag{8.1}$$

Let $\eta > 0$ be given. For all $\delta > 0$ it holds

$$\begin{aligned}
&\sup_{\vartheta\in\theta} \mathbb{P}(Z_{\vartheta,n} X_{\vartheta,n} + Y_{\vartheta,n} \in B) \\
&\le \sup_{\vartheta\in\theta} \mathbb{P}(Z_{\vartheta,n} X_{\vartheta,n} \in B^{\delta}) + \sup_{\vartheta\in\theta} \mathbb{P}(|Y_{\vartheta,n}| \ge \delta),
\end{aligned}$$

for large n the second term is smaller than η owing to the assumptions on $Y_{\vartheta,n}$,

$$\le \sup_{\vartheta\in\theta} \mathbb{P}(Z_{\vartheta,n} X_{\vartheta,n} \in B^{\delta}) + \eta.$$

As a single random vector X is tight meaning that for each $\eta > 0$ there is M such that $\mathbb{P}(|X| \ge M) < \eta$. By taking the set $\{x \,|\, x \ge M\}$ in the definition of uniform convergence in total variation we obtain for n large enough $\sup_{\vartheta\in\theta} \mathbb{P}(|X_{\vartheta,n}| \ge M) < 2\eta$. By considering possibly larger n we can also ensure $\sup_{\vartheta\in\theta} \mathbb{P}(|Z_{\vartheta,n} - 1| \ge \delta/M) \le \eta$. But if $|X_{\vartheta,n}| < M$ and $|Z_{\vartheta,n} - 1| < \delta/M$, then $|X_{\vartheta,n} Z_{\vartheta,n} - X_{\vartheta,n}| < \delta$. For n large enough it holds

$$\begin{aligned}
\sup_{\vartheta\in\theta} \mathbb{P}(Z_{\vartheta,n} X_{\vartheta,n} + Y_{\vartheta,n} \in B) &\le \sup_{\vartheta\in\theta} \mathbb{P}(Z_{\vartheta,n} X_{\vartheta,n} \in B^{\delta}) + \eta \\
&\le \sup_{\vartheta\in\theta} \mathbb{P}(X_{\vartheta,n} \in B^{2\delta}) + 4\eta \\
&\le \mathbb{P}(X \in B^{2\delta}) + 5\eta \\
&\le \mathbb{P}(X \in B) + 6\eta, \qquad (8.2)
\end{aligned}$$

for δ small enough by (8.1).

Let $\eta > 0$ be given. For n large enough and $\delta > 0$ small enough we obtain similarly

$$\begin{aligned}
\inf_{\vartheta\in\theta} \mathbb{P}(Z_{\vartheta,n} X_{\vartheta,n} + Y_{\vartheta,n} \in B) &\ge \inf_{\vartheta\in\theta} \mathbb{P}(Z_{\vartheta,n} X_{\vartheta,n} \in B^{-\delta}) - \eta \\
&\ge \inf_{\vartheta\in\theta} \mathbb{P}(X_{\vartheta,n} \in B^{-2\delta}) - 4\eta \\
&\ge \mathbb{P}(X \in B^{-2\delta}) - 5\eta \\
&\ge \mathbb{P}(X \in B) - 6\eta. \qquad (8.3)
\end{aligned}$$

The statement follows by combining (8.2) and (8.3). □

Lemma 8 outlines how to proceed in showing uniform convergence. $X_{\vartheta,n}$ will be the leading term of the linearized stochastic error, $Y_{\vartheta,n}$ will be the sum

of the smaller stochastic errors, the remainder term and the approximation error and $Z_{\vartheta,n}$ will be the quotient of standard deviation and estimated standard deviation. The approximation error is uniformly controlled over $\mathcal{G}_s(R, \sigma_{\max})$ and thus over $\mathcal{H}_s(R, \sigma_{\max})$, too.

The substitution of the standard deviation by its empirical counterpart works as follows. We fix some x and write d instead of $d(x)$ in Theorem 1 to unify the notation with Theorem 3 and to treat both simultaneously. For δ continuous and positive the standard deviation depends continuously on γ and λ through d. γ and λ are restricted to a compact set. By the uniform convergence and by an upper bound of the standard deviation we obtain $\sup_{\mathcal{T}} \mathbb{Q}_{\mathcal{T}}(|\Delta\hat{\rho}| > \delta) \to 0$ for all $\delta > 0$ and for $\rho \in \{\gamma, \lambda\}$, where the supremum is over all $\mathcal{T} \in \mathcal{H}_s(R, \sigma_{\max})$ with a fixed volatility σ. Since d is uniformly continuous in γ and λ, we obtain $\sup_{\mathcal{T}} \mathbb{Q}_{\mathcal{T}}(|\Delta\hat{d}| > \delta) \to 0$ for all $\delta > 0$, which gives the assumption on $d/\hat{d}$ corresponding to $Z_{\vartheta,n}$ in Lemma 8 by a lower bound on d.

By the following lemma uniform convergence in total variation of the linearized stochastic error follows from uniform convergence in each component of the covariance matrix.

Lemma 9. *Let X be a normal random vector with symmetric positive definite covariance matrix $A \in \mathbb{R}^{d\times d}$. Let $X_{\vartheta,n}$, $n \in \mathbb{N}$, $\vartheta \in \theta$, be normal random vectors with covariance matrices $A_{\vartheta,n} \in \mathbb{R}^{d\times d}$. If $A_{\vartheta,n}$ converge to A in each component uniformly over θ as $n \to \infty$, then $X_{\vartheta,n}$ converge to X in total variation uniformly over θ as $n \to \infty$.*

Proof. We have to show that

$$\sup_{\vartheta\in\theta} \sup_{B} \left| \int_B \frac{\exp(-\langle x, A_{\vartheta,n}^{-1}x\rangle/2)}{\det(\sqrt{2\pi}A_{\vartheta,n})} - \frac{\exp(-\langle x, A^{-1}x\rangle/2)}{\det(\sqrt{2\pi}A)} \mathrm{d}x \right| \tag{8.4}$$

converges to zero as $n \to \infty$. The determinant is a continuous function of the components of a matrix. For all $\delta \in (0, \det(\sqrt{2\pi}A))$ there is $N \in \mathbb{N}$ such that $\sup_{\vartheta\in\theta} |\det(\sqrt{2\pi}A_{\vartheta,n}) - \det(\sqrt{2\pi}A)| \le \delta$ holds for all $n \ge N$ and $A_{\vartheta,n}^{-1}$ is well-defined for all $\vartheta \in \theta$ if $n \ge N$. Expression (8.4) equals

$$\begin{aligned} &= \sup_{\vartheta\in\theta} \frac{1}{2} \int_{\mathbb{R}^d} \left| \frac{\exp(-\langle x, A_{\vartheta,n}^{-1}x\rangle/2)}{\det(\sqrt{2\pi}A_{\vartheta,n})} - \frac{\exp(-\langle x, A^{-1}x\rangle/2)}{\det(\sqrt{2\pi}A)} \right| \mathrm{d}x \\ &= \sup_{\vartheta\in\theta} \frac{1}{2} \int_{\mathbb{R}^d} \frac{\exp(-\langle x, A^{-1}x\rangle/2)}{|\det(\sqrt{2\pi}A)|} \\ &\qquad \left| \frac{\det(\sqrt{2\pi}A)\exp(-\langle x, (A_{\vartheta,n}^{-1} - A^{-1})x\rangle/2)}{\det(\sqrt{2\pi}A_{\vartheta,n})} - 1 \right| \mathrm{d}x. \end{aligned} \tag{8.5}$$

$A_{\vartheta,n}$ converges to A in each component uniformly over θ. Likewise $A_{\vartheta,n}^{-1}$ converges to A^{-1} in each component uniformly over θ. We now addition-

ally require $\delta \in (0, \lambda_{\min}(A^{-1})/d)$, where $\lambda_{\min}(A^{-1})$ is the smallest eigenvalue of A^{-1}. By going over to a possibly larger N we may assume that $\sup_{\vartheta\in\theta} |(A_{\vartheta,n}^{-1} - A^{-1})_{jk}| \leq \delta$ for all $j, k = 1, \ldots, d$, for all $n \geq N$. Then for all $n \geq N$

$$
\begin{aligned}
|\langle x, (A_{\vartheta,n}^{-1} - A^{-1})x\rangle| &= \left| \sum_{j,k=1}^{d} x_j (A_{\vartheta,n}^{-1} - A^{-1})_{jk} x_k \right| \\
&\leq \delta \left(\sum_{j=1}^{d} |x_j| \right)^2 \leq \delta d \|x\|_2^2,
\end{aligned}
$$

where in the last step the Cauchy-Schwarz inequality is used. We see that (8.5) can be bounded by

$$
\frac{1}{2} \int_{\mathbb{R}^d} \frac{\exp(-\langle x, A^{-1}x\rangle/2)}{|\det(\sqrt{2\pi}A)|}
\left(\frac{|\det(\sqrt{2\pi}A)| \exp(d\delta\|x\|_2^2/2)}{|\det(\sqrt{2\pi}A)| - \delta} - \frac{|\det(\sqrt{2\pi}A)| \exp(-d\delta\|x\|_2^2/2)}{|\det(\sqrt{2\pi}A)| + \delta} \right) \mathrm{d}x. \tag{8.6}
$$

The integrand converges pointwise to zero for $\delta \to 0$. For fixed δ the function is at the same time a dominating function, which is integrable since $A^{-1} - \delta d I_d$ is positive definite by the choice $\delta \in (0, \lambda_{\min}(A^{-1})/d)$. By the dominated convergence theorem (8.6) converges to zero and likewise (8.4). $\square$

We will show uniform convergence in each component of the covariance matrix. By Lemma 9 this leads to uniform convergence in total variation of the linearized stochastic errors provided that the covariance matrix of the limit is nondegenerated. To this end we assume in the case $\sigma = 0$ that δ is positive and that $\int_0^1 u^4 w_\rho^1(u)^2 \mathrm{d}u > 0$ for the weight functions w_ρ^1, $\rho \in \{\sigma, \gamma, \lambda, 0, \mu\}$, involved. Joint distributions may further only involve more than one of the estimators $\hat{\sigma}^2$, $\hat{\lambda}$ or $\hat{\mu}(0)$, if the covariance matrix is positive definite. in the case $\sigma > 0$ we assume that $w_\sigma^1(1), w_\gamma^1(1), w_\lambda^1(1), w_\mu^1(1) \neq 0$ and $\|\delta^2\|_{L^2(\mathbb{R})} > 0$. By the uniform version of Lemma 5 the remainder term converges uniformly to zero. By Lemma 8 each of the rescaled stochastic error terms

$$
\begin{aligned}
&\frac{1}{\epsilon \exp(T\sigma^2 U^2/2) w_\sigma^1(1)} 2 \int_0^1 \mathrm{Re}(\Delta\psi_\epsilon(Uu)) w_\sigma^1(u) \mathrm{d}u, \\
&\frac{1}{\epsilon \exp(T\sigma^2 U^2/2) w_\gamma^1(1)} 2 \int_0^1 \mathrm{Im}(\Delta\psi_\epsilon(Uu)) w_\gamma^1(u) \mathrm{d}u, \\
&\frac{1}{\epsilon \exp(T\sigma^2 U^2/2) w_\lambda^1(1)} 2 \int_0^1 \mathrm{Re}(\Delta\psi_\epsilon(Uu)) w_\lambda^1(u) \mathrm{d}u,
\end{aligned}
$$

$$\frac{2\pi}{\epsilon\exp(T\sigma^2U^2/2)w_\mu^1(1)}\mathcal{F}^{-1}\left[\Delta\psi_\epsilon(Uu)w_\mu^1(u)\right](Ux)$$

converges uniformly in distribution to

$$\frac{\sqrt{2}\|\delta^2\|_{L^2(\mathbb{R})}}{\exp(T(\sigma^2/2+\gamma-\lambda))T^2\sigma^2}W,$$

where W is a standard normal random variable. By the uniform versions of Lemma 3 and Lemma 5 we also obtain that for the second and third of the above stochastic error terms holds joint uniform convergence in distribution to independent normal variables.

8.1 Uniform convergence in the case $\sigma = 0$

The convergence in distribution of the linearized stochastic errors is shown in Lemma 1 by the convergence of the components in the covariance matrix. We restrict ourselves to the case $x_1 = \cdots = x_n$ and show uniform convergence in each component of the covariance matrix. This implies uniform convergence in total variation by Lemma 9. We assume that δ is continuous at all $x \in [x_1 - TR, x_1 + TR]$. We note that f_U, g_U and θ_j depend on the Lévy triplet $\mathcal{T}$. The uniform convergence of the rescaled covariances (6.12) will be shown by the following easy lemma.

Lemma 10. *Let $f_{\vartheta,n}, f_\vartheta, g_{\vartheta,n}, g_\vartheta \in L^1([0,1],\mathbb{C})$, $n \in \mathbb{N}$, $\vartheta \in \theta$, and $M > 0$ such that $\|f_{\vartheta,n}\|_\infty, \|g_\vartheta\|_\infty \leq M$ for all $n \in \mathbb{N}$, for all $\vartheta \in \theta$. Let $\sup_{\vartheta\in\theta}\|f_{\vartheta,n} - f_\vartheta\|_{L^1([0,1],\mathbb{C})} \to 0$ and $\sup_{\vartheta\in\theta}\|g_{\vartheta,n} - g_\vartheta\|_{L^1([0,1],\mathbb{C})} \to 0$ as $n \to \infty$. Then*

$$\sup_{\vartheta\in\theta}\int_0^1 |f_{\vartheta,n}(x)g_{\vartheta,n}(x) - f_\vartheta(x)g_\vartheta(x)|\mathrm{d}x \to 0 \qquad \text{as } n \to \infty.$$

Proof. For all $\vartheta \in \theta$ it holds

$$\begin{aligned}|f_{\vartheta,n}g_{\vartheta,n} - f_\vartheta g_\vartheta| &\leq |f_{\vartheta,n}g_{\vartheta,n} - f_{\vartheta,n}g_\vartheta| + |f_{\vartheta,n}g_\vartheta - f_\vartheta g_\vartheta| \\ &\leq M|g_{\vartheta,n} - g_\vartheta| + M|f_{\vartheta,n} - f_\vartheta|.\end{aligned}$$

By assumption $\sup_{\vartheta\in\theta}\|f_{\vartheta,n} - f_\vartheta\|_{L^1([0,1],\mathbb{C})} \to 0$ as $n \to \infty$ and $\sup_{\vartheta\in\theta}\|g_{\vartheta,n} - g_\vartheta\|_{L^1([0,1],\mathbb{C})} \to 0$ as $n \to \infty$ and the claimed statement follows. □

Let us verify the assumptions of Lemma 10 for the first factor in (6.12). f_U and $u^2w_j(u)$ will correspond to $f_{\vartheta,n}$ and f_ϑ, respectively. It holds

$$\begin{aligned}&|f_U(u) + u^2w_j(u)| \\ &= |u^2w_j(u)(1 - \exp(-T\mathcal{F}\mu(Uu))) + iu\exp(-T\mathcal{F}\mu(Uu))/U| \\ &\leq u^2|w_j(u)|(\exp(TR(1\wedge h(Uu))) - 1) + iu\exp(TR)/U. \qquad (8.7)\end{aligned}$$

This bound does not depend on $\mathcal{T}$ and converges everywhere to zero. Further $f_U(u), u^2 w_j(u) \leq \sqrt{2}\|w_j\|_\infty \exp(TR)$ is a bound that does not depend on U nor on $\mathcal{T}$. By the dominated convergence theorem

$$\sup_{\mathcal{T}} \|f_U(u) + u^2 w_j(u)\|_{L^1([0,1],\mathbb{C})} \to 0 \tag{8.8}$$

as $U \to \infty$ and the conditions on the first factor in Lemma 10 are satisfied.

To show the assumptions of Lemma 10 on the second factor, which is the complex conjugate of

$$\begin{aligned}&(g_U(v) * \mathcal{F}^{-1}(\delta(y/U + \theta_j)^2)(v))(u)\\ =& \int_{-\infty}^{\infty} g_U(u-v)\mathcal{F}^{-1}(\delta(y+\theta_j)^2)(Uv)U\mathrm{d}v,\end{aligned}$$

we apply the following lemma. It is a uniform version of the basic theorem on approximate identities.

Lemma 11. *Let $f, f_{\vartheta,n} \in L^1(\mathbb{R},\mathbb{C})$, $n \in \mathbb{N}$, $\vartheta \in \theta$. Let $\sup_{\vartheta\in\theta} \|f_{\vartheta,n} - f\|_{L^1(\mathbb{R},\mathbb{C})} \to 0$ for $n \to \infty$. Let $\delta_{\vartheta,n} \in L^1(\mathbb{R},\mathbb{C})$ fulfill the following properties:*

(i) There exists $c > 0$ such that $\|\delta_{\vartheta,n}\|_{L^1(\mathbb{R},\mathbb{C})} \leq c$ for all $n \in \mathbb{N}$, for all $\vartheta \in \theta$.

(ii) $\int_{-\infty}^{\infty} \delta_{\vartheta,n}(y)\mathrm{d}y = c_\vartheta$ for all $n \in \mathbb{N}$, for all $\vartheta \in \theta$.

(iii) For any neighborhood V of zero we have $\sup_{\vartheta\in\theta} \int_{V^c} |\delta_{\vartheta,n}(y)|\mathrm{d}y \to 0$ as $n \to \infty$.

Then $\sup_{\vartheta\in\theta} \|\delta_{\vartheta,n} * f_{\vartheta,n} - c_\vartheta f\|_{L^1(\mathbb{R},\mathbb{C})} \to 0$ as $n \to \infty$.

Proof. By the triangle inequality

$$\begin{aligned}&\sup_{\vartheta\in\theta} \|\delta_{\vartheta,n} * f_{\vartheta,n} - c_\vartheta f\|_{L^1(\mathbb{R},\mathbb{C})}\\ &\leq \sup_{\vartheta\in\theta} \|\delta_{\vartheta,n} * (f_{\vartheta,n} - f)\|_{L^1(\mathbb{R},\mathbb{C})} + \sup_{\vartheta\in\theta} \|\delta_{\vartheta,n} * f - c_\vartheta f\|_{L^1(\mathbb{R},\mathbb{C})}.\end{aligned} \tag{8.9}$$

The first term in the triangle inequality (8.9) can be bounded by

$$\begin{aligned}\sup_{\vartheta\in\theta} \|\delta_{\vartheta,n} * (f_{\vartheta,n} - f)\|_{L^1(\mathbb{R},\mathbb{C})} &\leq \sup_{\vartheta\in\theta} \|\delta_{\vartheta,n}\|_{L^1(\mathbb{R},\mathbb{C})} \, \|f_{\vartheta,n} - f\|_{L^1(\mathbb{R},\mathbb{C})}\\ &\leq c \sup_{\vartheta\in\theta} \|f_{\vartheta,n} - f\|_{L^1(\mathbb{R},\mathbb{C})},\end{aligned}$$

which converges to zero by assumption. To bound the second term in the triangle inequality (8.9) proceed as in the proof of Theorem 1.2.19 in [13, p. 26]. Without loss of generality we may assume that $\|f\|_{L^1(\mathbb{R},\mathbb{C})} > 0$. Continuous functions with compact support are dense in $L^1(\mathbb{R},\mathbb{C})$. Since

continuous functions g with compact support are bounded we obtain by the dominated convergence theorem

$$\int_{-\infty}^{\infty} |g(x-y) - g(x)| \mathrm{d}x \to 0$$

as $y \to 0$. We approximate $f \in L^1(\mathbb{R}, \mathbb{C})$ by a continuous function with compact support and see that for all $\delta > 0$ there is some neighborhood V of zero such that

$$\int_{-\infty}^{\infty} |f(x-y) - f(x)| \mathrm{d}x < \frac{\delta}{2c} \qquad \text{for all } y \in V. \tag{8.10}$$

It further holds

$$\begin{aligned}
&(\delta_{\vartheta,n} * f)(x) - c_\vartheta f(x) \\
&= (\delta_{\vartheta,n} * f)(x) - f(x) \int_{-\infty}^{\infty} \delta_{\vartheta,n}(y) \mathrm{d}y \\
&= \int_{-\infty}^{\infty} (f(x-y) - f(x)) \delta_{\vartheta,n}(y) \mathrm{d}y \\
&= \int_V (f(x-y) - f(x)) \delta_{\vartheta,n}(y) \mathrm{d}y + \int_{V^c} (f(x-y) - f(x)) \delta_{\vartheta,n}(y) \mathrm{d}y.
\end{aligned}$$

We take L^1 norms with respect to x and obtain for the first term by (8.10)

$$\begin{aligned}
&\sup_{\vartheta \in \theta} \left\| \int_V (f(x-y) - f(x)) \delta_{\vartheta,n}(y) \mathrm{d}y \right\|_{L^1(\mathbb{R},\mathbb{C})} \\
&\leq \sup_{\vartheta \in \theta} \int_V \|f(x-y) - f(x)\|_{L^1(\mathbb{R},\mathbb{C})} |\delta_{\vartheta,n}(y)| \mathrm{d}y \\
&\leq \sup_{\vartheta \in \theta} \int_V \frac{\delta}{2c} |\delta_{\vartheta,n}(y)| \mathrm{d}y < \frac{\delta}{2}
\end{aligned} \tag{8.11}$$

and for the second term

$$\begin{aligned}
&\sup_{\vartheta \in \theta} \left\| \int_{V^c} (f(x-y) - f(x)) \delta_{\vartheta,n}(y) \mathrm{d}y \right\|_{L^1(\mathbb{R},\mathbb{C})} \\
&\leq \sup_{\vartheta \theta} \int_{V^c} 2\|f\|_{L^1(\mathbb{R},\mathbb{C})} |\delta_{\vartheta,n}(y)| \mathrm{d}y < \frac{\delta}{2},
\end{aligned} \tag{8.12}$$

where n is taken large enough such that

$$\sup_{\vartheta \in \theta} \int_{V^c} |\delta_{\vartheta,n}(y)| \mathrm{d}y < \frac{\delta}{4\|f\|_{L^1(\mathbb{R},\mathbb{C})}}.$$

The lemma is a consequence of (8.11) and (8.12). □

Let us first verify the assumptions (i), (ii) and (iii) for $\mathcal{F}^{-1}(\delta(y+\theta_j)^2(Uv)U$, which will correspond to $\delta_{\vartheta,n}$. We have

$$\mathcal{F}^{-1}(\delta(y+\theta)^2)(v) = e^{i\theta v}\mathcal{F}^{-1}(\delta(y)^2)(v). \tag{8.13}$$

By the assumptions of Lemma 1 it holds $\mathcal{F}\delta^2 \in L^1(\mathbb{R})$. The equality

$$|\mathcal{F}^{-1}(\delta(y+\theta_j)^2)(Uv)U| = |\mathcal{F}^{-1}(\delta(y)^2)(Uv)U|$$

shows that the absolute value does not depend on $\mathcal{T}$ and that conditions (i) and (iii) are satisfied. Condition (ii) is satisfied by (6.13). We have

$$\sup_{\mathcal{T}} \|g_U(u) + u^2 w_k(u)\|_{L^1([0,1],\mathbb{C})} \to 0 \tag{8.14}$$

as in the corresponding equation (8.8) for f_U. By extending g_U and w_k by zero outside $[0,1]$ this holds in $L^1(\mathbb{R},\mathbb{C})$, too. We apply Lemma 11 to $g_U(v)$ and $v^2 w_k(v)$, which correspond to $f_{\vartheta,n}$ and f in this lemma. $v^2 w_k(v)$ is bounded and since condition (i) is satisfied we see that $\delta(\theta_j)^2 v^2 w_k(v)$ is uniformly bounded over all Lévy triplets. By Lemma 10 the convergence of the covariances (6.14) holds uniformly. The convergence in the analogous equation without conjugation (6.15) holds uniformly, too.

8.2 Uniform convergence in the case $\sigma > 0$

Let us first fix $\sigma > 0$ and prove uniform convergence for all Lévy triplets with this fixed value of σ. To this end we show uniform convergence in Lemma 2. In order to control the error when going over to smaller domain of integration in (6.22) the term $\overline{\mathcal{F}^{-1}(\delta(y+\theta)^2)(U(u-v))} f_U(u)\overline{g_U(v)}/(uv)$ needs to be bounded uniformly. The inverse Fourier transform is bounded by the L^1-norm of δ^2. The functions f_U and g_U are uniformly bounded since $\|\mathcal{F}\mu\|_\infty \leq R$. The crucial step is the limit (6.23), where the refined dirac sequence argument is applied. As stated in (8.13), a translation before the Fourier transform is equal to a multiplication by a complex unit after the Fourier transform. Since $|\theta| \leq T(\sigma_{\max}^2 + R)$ the complex unit tends uniformly to one. w_U and $\tilde{w}_U$ do not depend on the Lévy triplet. Since there is h with $|\mathcal{F}\mu(u)| \leq R\, h(u)$ and $h(u) \to 0$ as $|u| \to \infty$ the factor $(-u + i/U)/\exp(T\mathcal{F}\mu(Uu))$ converges to $-u$ uniformly over $\mathcal{H}_s(R, \sigma_{\max})$. This leads to uniform convergence in (6.23) and thus in (6.24). To see that the covariance without conjugation converges in (6.25) uniformly to zero we observe that $\mathcal{F}^{-1}(\delta(y+\theta)^2)(u) \to 0$ for $|u| \to \infty$ uniformly since the translation by θ corresponds to a multiplication by a complex unit by equation (8.13). We immediately obtain the uniform convergence in each component of the covariance matrix in Lemma 3.

8.3 Uniform convergence of the remainder term

To begin with, we observe that Proposition 1 holds with the same constant for all Lévy triplets in $\mathcal{H}_s(R, \sigma_{\max})$. This follows from the bound (6.7) in the proof and from the fact that $X(u) = \int_{-\infty}^{\infty} e^{iux}\delta(x)\mathrm{d}W(x)$ does not depend on the Lévy triplet.

Lemma 5 and Lemma 6 hold uniformly, meaning that for all $\eta > 0$

$$\sup_{\mathcal{T}\in\mathcal{H}_s(R,\sigma_0)} \mathbb{Q}_{\mathcal{T}}\left(\left|\frac{1}{\epsilon\exp(T\sigma_0^2 U(\epsilon)^2/2)}\int_0^1 w_{U(\epsilon)}(u)\mathcal{R}_{\epsilon,U(\epsilon)}(u)\mathrm{d}u\right| > \eta\right) \to 0,$$
$$\sup_{\mathcal{T}\in\mathcal{H}_s(R,0)} \mathbb{Q}_{\mathcal{T}}\left(\left|\frac{1}{\epsilon U(\epsilon)^{3/2}}\int_0^1 w_{U(\epsilon)}(u)\mathcal{R}_{\epsilon,U(\epsilon)}(u)\mathrm{d}u\right| > \eta\right) \to 0,$$

as $\epsilon \to 0$. For Lemma 6 this follows from the uniform convergence of the bound in the proof. For Lemma 5 this can be seen by the corresponding uniform statements along the lines of the proof up to the bound (6.29). Then we bound (6.29) in two different ways depending on whether $\sigma^2 \in [0, \sigma_0^2/2]$ or $\sigma^2 \in (\sigma_0^2/2, \sigma_0^2]$. In the latter case we can proceed as in the proof for $\sigma > 0$. Since σ^2 is bounded from below by $\sigma_0^2/2 > 0$ the convergence is uniform. For $\sigma^2 \in [0, \sigma_0^2/2]$ we estimate

$$\begin{aligned}&\frac{C}{\epsilon\exp(T\sigma_0^2U^2/2)}\int_0^1 \frac{\epsilon^2(U^2+U^4)u\exp(T\sigma^2U^2u^2)\|\delta\|_{L^2(\mathbb{R})}^2}{T^2\exp(2T(\sigma^2/2+\gamma-\lambda)-2T\|\mathcal{F}\mu\|_\infty)}\mathrm{d}u\\ &\le \frac{C\epsilon(U^2+U^4)\|\delta\|_{L^2(\mathbb{R})}^2}{T^2\exp(2T(\sigma^2/2+\gamma-\lambda)-2T\|\mathcal{F}\mu\|_\infty)},\end{aligned}$$

which converges uniformly to zero by $\sigma_0 > 0$ and by the assumption

$$\epsilon U^2\sqrt{\log(U)}\exp(T\sigma_0^2U^2/2) \to 0$$

of Lemma 5. The maximum of the bounds is a bound that holds for all Lévy triplets in the class. This shows the uniform version of Lemma 5.

References

[1] Denis Belomestny. Spectral estimation of the fractional order of a Lévy process. *Ann. Statist.*, 38(1):317–351, 2010.

[2] Denis Belomestny and Markus Reiß. Spectral calibration of exponential Lévy models. *Finance and Stochastics*, 10(4):449–474, 2006.

[3] Denis Belomestny and Markus Reiß. Spectral calibration of exponential Lévy models [2]. Discussion Paper 2006-035, SFB 649, Humboldt-Universität zu Berlin, Germany, 2006. Available at http://sfb649.wiwi.hu-berlin.de/papers/pdf/SFB649DP2006-035.pdf.

[4] Lawrence D. Brown and Mark G. Low. Asymptotic equivalence of nonparametric regression and white noise. *Ann. Statist.*, 24(6):2384–2398, 1996.

[5] P. Carr and D. Madan. Option valuation using the fast Fourier transform. *Journal of Computational Finance*, 2(4):61–73, 1999.

[6] Rama Cont. Model uncertainty and its impact on the pricing of derivative instruments. *Math. Finance*, 16(3):519–547, 2006.

[7] Rama Cont and Peter Tankov. *Financial modelling with jump processes*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[8] Rama Cont and Peter Tankov. Non-parametric calibration of jump–diffusion option pricing models. *Journal of computational finance*, 7(3):1–50, 2004.

[9] Rama Cont and Peter Tankov. Retrieving Lévy processes from option prices: regularization of an ill-posed inverse problem. *SIAM J. Control Optim.*, 45(1):1–25 (electronic), 2006.

[10] D. Ellsberg. Risk, ambiguity, and the savage axioms. *The Quarterly Journal of Economics*, pages 643–669, 1961.

[11] Jianqing Fan. Asymptotic normality for deconvolution kernel density estimators. *Sankhyā Ser. A*, 53(1):97–110, 1991.

[12] J.E. Figueroa-López. Sieve-based confidence intervals and bands for Lévy densities. *Bernoulli*, 17(2):643–670, 2011.

[13] Loukas Grafakos. *Classical and modern Fourier analysis*. Pearson Education, Inc., Upper Saddle River, NJ, 2004.

[14] I. Grama and M. Nussbaum. Asymptotic equivalence for nonparametric regression. *Math. Methods Statist.*, 11(1):1–36, 2002.

[15] Jean-Pierre Kahane. *Some random series of functions*, volume 5 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1985.

[16] Olav Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[17] F.H. Knight. Risk, uncertainty and profit. *New York Houghton Mifflin*, 1921.

[18] R.C. Merton. Option pricing when underlying stock returns are discontinuous. *Journal of financial economics*, 3(1-2):125–144, 1976.

[19] Jakob Söhl. Polar sets for anisotropic Gaussian random fields. *Statistics & Probability Letters*, 80(9-10):840 – 847, 2010.

[20] Jakob Söhl and Mathias Trabs. Option calibration of exponential Lévy models: Implementation and empirical results. 2012. arXiv:1202.5983.

[21] Peter Tankov. Pricing and hedging in exponential Lévy models: review of recent results. In *Paris-Princeton Lectures on Mathematical Finance 2010*, volume 2003 of *Lecture Notes in Math.*, pages 319–359. Springer, Berlin, 2011.

[22] Mathias Trabs. Calibration of selfdecomposable Lévy models. Discussion Paper 2011-073, SFB 649, Humboldt Universität zu Berlin, Germany, 2011. Available at http://sfb649.wiwi.hu-berlin.de/papers/pdf/SFB649DP2011-073.pdf.

[23] A. J. van Es and H.-W. Uh. Asymptotic normality of nonparametric kernel type deconvolution density estimators: crossing the Cauchy boundary. *J. Nonparametr. Stat.*, 16(1-2):261–277, 2004.

[24] Bert van Es and Hae-Won Uh. Asymptotic normality of kernel-type deconvolution estimators. *Scand. J. Statist.*, 32(3):467–483, 2005.